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COMMON FIXED POINT THEOREM FOR E.A. PROPERTY IN *C**-ALGEBRA VALUED SYMMETRIC SPACES

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Abstract. This chapter consists a common fixed point theorem for weakly com patible self mappings satisfying E.A. property in the framework of C^*−algebra valued symmetric spaces.

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1. Introduction and preliminaries

The classical Banach contraction principle [1] has been generalized in many ways. One direction of the generalization of this principle can be referred as [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

Recall the following notations and definitions:

- $*$ denotes the linear involution.
- A is denoted by complex algebra $*$-algebra.
- 1_A is an unital element.
- (A, $*$) is denoted by an unital $*$-algebra, if it contains 1_A.
- (A, $*$) denotes the Banach $*$-algebra, if it satisfies $kp^*k = kpk$ and $kpqk \leq kpkkqk$.
- (A, $*$) is denoted by C^*-algebra if $kp^*pk = kpk^2$, for all $p \in A$.

Throughout the chapter, A is denoted by C^*-algebra with 1_A.

The class of C^*-algebra valued metric spaces is introduced by Ma et al. [4] in 2014 and utilize the same to prove fixed point results with an application. In 2020, Asim and Imdad [5] introduced the notion of C^*-algebra valued symmetric space to enlarge the class of C^*-algebra valued metric space given as follows:

Definition 1.1. [5] "*Suppose X 6= ∅. The mapping* $d : X \times X \to A$ *is known as* C^**-algebra valued symmetric on X, if* $\forall p, q \in X$:

(*i*) $d(p, q) < 0_A$ *and* $d(p, q) = 0_A$ *iff* $p = q$;
(*ii*) $d(p, q) = d(q, p)$.

The triplet (X, A, d) *is called a* C^**-algebra valued symmetric space.*"

Definition 1.2. [5] *Let* (X, A, d) *is* C^**-algebra valued symmetric space and a sequence* $\{p_n\}$ *in X. Then*

- $\{p_n\}$ *is convergent to* $p \in A$ *if*

$$\lim_{n\to\infty} d(p_n, p) = 0_A.$$

- $\{p_n\}$ *is Cauchy if*

$$\lim_{n,m\to\infty} d(p_n, p_m) = 0_{\mathbb{A}}.$$

- $(A, \mathbb{A}, d)$ *is complete if each Cauchy sequence in A is convergent to a point* $p \in X$.

2. Main results

Before giving our main result we state the following lemma which will be utilized in the following sequel:

Lemma 2.1. *Let* f_1, f_2, f_3 *and* f_4 *be four self maps of a* C^**-algebra-valued symmetric space* $(X, \mathbb{A}, d)$ *satisfying the followings:*

(Li) $\{f_1, f_3\}$ *(or* $\{f_2, f_4\}$*) satisfies (E.A.) property,*

(Lii) $f_3(X) \subset f_2(X)$ *(or* $f_4(X) \subset f_1(X)$*),*

(Liii) f_1, f_2, f_3 *and* f_4 *satisfy*

$$f_3X \subseteq f_2X,\ f_4X \subseteq f_1X,$$

$$d(f_3p, f_4q) \preccurlyeq a^*m(p, q)a, \quad (1)$$

for any $p, q \in X$, *where* $a \in \mathbb{A}$ *with* $\|a\| < 1$ *and*

$$m(p, q) = \max\{d(f_1p, f_2q), d(f_3p, f_1p), d(f_4p, f_2q), d(f_3p, f_2q), d(f_4q, f_1p)\}.$$

Then $\{f_1, f_3\}$ *and* $\{f_2, f_4\}$ *satisfy (E.A.) common property.*

Proof. If the pair $\{f_1, f_3\}$ satisfies the (E.A.) property, then there exists a sequence $\{p_n\}$ in X such that

$$\lim_{n\to\infty} f_1p_n = \lim_{n\to\infty} f_3p_n = r, \text{ for some } r \in X.$$

Since $f_3(X) \subset f_2(X)$, hence for each $\{p_n\}$ there exists $\{q_n\}$ in X such that $f_3p_n = f_2q_n$. Therefore, $\lim_{n\to\infty} f_2q_n = \lim_{n\to\infty} f_3p_n = r$, for some $r \in X$. Thus, we have that $f_2q_n \to r$, $f_3p_n \to r$ and $f_1p_n \to r$. Now , we assert that $T q_n \to r$. Let on contrary that $f_4q_n \to t \neq r$, then from (1), we have

$$d(f_3p_n, f_4q_n) \preccurlyeq a^*m(p_n, q_n)a.$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} d(f_3p_n, f_4q_n) = \lim_{n\to\infty} d(r, t) \preccurlyeq \lim_{n\to\infty} a^*m(p_n, q_n)a = a^*\lim_{n\to\infty} d(t, r)a \quad (2)$$

where,

$$\lim_{n\to\infty} m(p_n, q_n) = \lim_{n\to\infty} \max\{d(f_1p_n, f_2q_n), d(f_3p_n, f_1p_n), d(f_4q_n, f_2q_n), d(f_3p_n, f_2q_n), d(f_4q_n, f_1p_n)\}$$
$$= \max\{0_{\mathbb{A}}, 0_{\mathbb{A}}, d(t, r), 0_{\mathbb{A}}, d(t, r)\} = \lim_{n\to\infty} d(t, r).$$

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Now, form condition (1) we obtain $d(r, t) \preccurlyeq a^*d(r, t)a$, with $\|a\| < 1$, implies

$$\|d(r, t)\| \le \|a\|^2\|d(r, t)\| < \|d(r, t)\|,$$

a contradiction. Therefore, $\lim_{n\to\infty} f_4q_n = r$ which shows that the pairs $\{f_1, f_3\}$ and $\{f_2, f_4\}$ satisfy (E.A.) common property.

Now we present our main result as under:

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Theorem 2.1. *Let f_1, f_2, f_3 and f_4 be four self mappings of a C*-algebra-valued symmetric space* (X, A, d) *satisfying* (2) *and the followings:*

(i) pairs (f_1, f_3) *and* (f_2, f_4) *are weakly compatible,*
(ii) pairs pairs (f_1, f_3) *and* (f_2, f_4) *satisfy the E.A. common property, (iii)* *f_1X and f_2X are closed subsets of X.*

Then f_1, f_2, f_3 and f_4 have a unique common fixed point.

Proof. Since the pairs (f_1, f_3) and (f_2, f_4) satisfy the E.A. common property. Then there exist two sequences $\{p_n\}$ and $\{q_n\}$ in X such that

$\lim_{n\to\infty} f_1p_n = \lim_{n\to\infty} f_3p_n = \lim_{n\to\infty} f_2q_n = \lim_{n\to\infty} f_4q_n = r$, for some $r \in X$. If fX is a closed subset of X, then $\lim_{n\to\infty} f_1p_n = r \in f_2X$. Therefore, there exists a point $s \in X$ such that $r = f_1s$. Now, we shall show that $f_1s = f_3s$. Let, if possible, f_1s 6= f_3s. Using (1), we have

$$d(f_3s, f_4q_n) \; 4 \; a^*m(s, q_n)a. \quad (3)$$

Letting $n \to \infty$, we have $\lim_{n\to\infty} d(f_3v, f_4q_n) \; 4 \; a^*\lim_{n\to\infty} m(s, q_n)a$, where $\lim_{n\to\infty} m(s, q_n)$

$$= \lim_{n\to\infty} \max \; d(f_1v, f_2q_n), d(f_3v, f_1v), d(f_4q_n, f_2q_n), (4)\; d(f_3v, f_2q_n), d(f_4q_n, f_1v)$$
$$= \max\{d(r, r), d(f_3s, r), d(r, r), d(f_3s, r), d(r, r)\}$$
$$= d(f_3s, r). \quad (5)$$

Letting $n \to \infty$ in (3) and using (4), we get

$$d(f_3s, r) \; 4 \; a^*d(f_3s, r)a,$$

with $kak < 1$, implies

$$kd(f_3s, r)k \le kak^2kd(f_3s, r)k < kd(f_3s, r)k,$$

a contradiction. Hence, $f_3s = r = f_1s$. Therefore, s is a coincidence point of the pair (f_1, f_3). If f_2X is a closed subset of X, then $\lim_{n\to\infty} f_2q_n = r \in f_2X$. Therefore, there exists a point $u \in X$ such that $r = f_2u$. Now, we shall show that $f_2u = f_4u$. Let, if possible, f_2u 6= f_4u. Using (1), we have

$$d(f_3p_n, f_4u) \; 4 \; a^*m(p_n, u)a. \quad (6)$$

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Taking $n \to \infty$, we have $\lim_{n\to\infty} d(f_3p_n, f_4u) \; 4 \; a^*\lim_{n\to\infty} m(p_n, u)a$, where $\lim_{n\to\infty} m(p_n, u)$

$$= \lim_{n\to\infty} \max \; d(f_1p_n, f_2u), d(f_3p_n, f_1p_n), (7)\; d(f_4u, f_2u), d(f_3p_n, f_2u), d(f_4u, f_1p_n)$$
$$= \max\{d(r, r), d(r, r), d(f_4u, r), d(r, r), d(f_4u, r)\}$$
$$= \max\{0_A, 0_A, d(f_4u, r), d(f_4u, r)\}$$
$$= d(f_4u, r). \quad (8)$$

Taking $n \to \infty$ in (6) and employing (7), we get

$$d(r, f_4u) \; 4 \; a^*d(r, f_4u)a,$$

with $kak < 1$, implies

$$kd(r, f_4u)k \le kak^2kd(r, f_4u)k < kd(r, f_4u)k,$$

a contradiction. Hence, $f_4u = r = f_2u$. Therefore, u is a coincidence point of the pair (f_2, f_4). Since the pair (f_2, f_4) is weakly compatible, therefore, $f_2f_4u = f_4f_2u$ which implies that $f_4f_4u = f_4f_2u = f_2f_4u = f_2f_2u$. Since $f_4X \subseteq f_1X$, there exists $s \in X$, such that, $f_4u = f_1s$.

$$d(f_3s, f_4u)\ 4\ a^*m(s, u)a \quad (9)$$

where

$$m(s, u) = \max \{d(f_1s, f_2u), d(f_3s, f_1s), d(f_4u, f_2u), d(f_3s, f_2u), d(f_4u, f_1s)\} = d(f_3s, f_1s) = d(f_3s, f_4u).$$

Therefore, from (9), we get

$$d(f_3s, f_4u)\ 4\ a^*d(f_3s, f_4u)a,$$

with $kak < 1$ implies

$$kd(f_3s, f_4u)k \leq kak^2kd(f_3s, f_4u)k < kd(f_3s, f_4u)k,$$

a contradiction. Therefore, $f_3s = f_4u = f_1s$. Thus, we have $f_4u = f_2u = f_3s = f_1s$. The weak compatibility of the pair (f_1, f_3) implies that $f_1f_3s = f_3f_1s = f_3f_3s = f_1f_1s$. Now, we claim that f_4u is the common fixed point of f_1, f_2, f_3 and f_4. Suppose that $f_4f_4u\ 6= f_4u$. From (1), we have

$$d(f_4u, f_4f_4u) = d(f_3s, f_4f_4u)$$

$$4\ a^*m(s, tu)a, \quad (10)$$

where

$$m(s, u) = \max \{d(f_1v, f_2f_4u), d(f_3s, f_1s), d(f_2f_4u, f_4f_4u), d(f_3s, f_2f_4u), d(f_4f_4u, f_1v)\} = \max\{d(f_4u, f_4f_4u), 0_A, 0_A, d(f_4u, f_4f_4u)\} = d(f_4u, f_4f_4u).$$

Using this value in (10) and $kak < 1$, we get

$$kd(f_4u, f_4f_4u)k \leq kak^2kd(f_4u, f_4f_4u)k < kd(f_4u, f_4f_4u)k,$$

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a contradiction. Hence, $f_4u = f_4f_4u = f_2f_4u$. Thus, f_4u is the common fixed point of f_2 and f_4. Similarly, we can prove that f_3s is the common fixed point of f_1 and f_3. Since $f_4u = f_3s$, f_4u is the common fixed point of f_1, f_2, f_3 and f_4.

Finally, we show that there exists a unique common fixed point of f_1, f_2, f_3 and f_4, suppose that, p and q be two common fixed points such that $p\ 6= q$. From (1), we have

$$d(f_3p, f_4q)\ 4\ a^*m(p, q)a, \quad (11)$$

where

$$m(p, q) = \max \{d(f_1p, f_2q), d(f_3p, f_1p), d(f_4q, f_2q), d(f_3p, f_2q), d(f_4q, f_1p)\} = \max\{d(p, q), 0_A, 0_A, d(p, q)\} = d(p, q).$$

By utilizing this value, the condition (11) with $kak < 1$ implies

$$kd(p, q)k \leq kak^2kd(p, q)k < kd(p, q)k,$$

which is a contradiction. Hence, p and q are equal which completes the proof.

Theorem 2.2. *The conclusions of Theorem 2.1 remain true if the condition (iii) is replaced by the following*

(iii)' $f_3X \subset f_2X$ *and* $f_4X \subset f_1X$.

Corollary 2.1. *The conclusions of Theorem 2.1 and 2.2 remain true if the conditions (iii) and (iii)' are replaced by the following*

(iii)" f_3X *and* f_4X *are closed subsets of* X *provided* $f_3X \subset f_2X$ *and* $f_4X \subset f_1X$.

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