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The α - Laplace Elzaki Integral Transform on Time Scales

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Abstract

This paper introduces the $\boldsymbol{\alpha}$-Laplace Elzaki transform, a significant generalization of the nabla version on time scales. This transform rigorously established the fundamental properties, including the existence theorem, linearity, and the convolution theorem. Additionally, we derive the key derivative characteristics that underpin its functionality. The $\boldsymbol{\alpha}$-Laplace Elzaki transform combines techniques for discrete, continuous, and hybrid systems and is a powerful approach for solving partial dynamic equations over time scales.

Keywords: Dynamic equations, Laplace Transform, Laplace-Elzaki transform, Time scales

AMS Classification: 26E70 , 44A35 , 26A33

1 Introduction

The integral transform is a mathematical method to study integral and differential problems. Research scholars have examined the theory and uses of transforms, such as the Sumudu, Hankel, Laplace, and Elzaki transforms. Any non-empty closed subset of $\mathbb{R}$ is called a time scale. Here, we assume that a time scale $\mathbb{T}$ is unbounded above and $t_0 \in \mathbb{T}$ is fixed. Forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined as $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. For $t \in \mathbb{T}$, is said to be right scattered if $\sigma(t) > t$, and it is left scattered if $\rho(t) < t$. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right dense point. If $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called the left dense point. Points that are right-scattered and left-scattered at the same time are called isolated. It means that $\rho(t) < t < \sigma(t)$. For $t \in \mathbb{T}$ the forward graininess function $\mu : \mathbb{T} \to [0, \infty)$ is $\mu(t) = \sigma(t) - t$. For $t \in \mathbb{T}$ the backward graininess function $\nu : \mathbb{T} \to [0, \infty)$ is $\nu(t) = t - \rho(t)$ [1, 2, 5, 6, 8, 11]. For a function $h : \mathbb{T} \to \mathbb{C}$, T. G. Thange and S. M. Chhatraband [10] have defined a new α-Laplace transform on time scales as

$$H(z) = \mathcal{L}^{\alpha}\{h(t)\} = \int_{t_0}^{\infty} e^{\rho}_{\ominus z^{\alpha}}(t, t_0)\, h(t)\, \nabla t.$$

In [7], authors have defined the Elzaki-Laplace Transform as

$$L_x E_t\{f(x, t)\} = \bar{f}(s, p) = p \int_0^{\infty}\int_0^{\infty} e^{-\left(sx+\frac{t}{p}\right)} f(x, t)\, dx\, dt.$$

This article defines the α-Laplace-Elzaki Transform on time scales to solve dynamic equations.

2 Preliminaries

All results in this section are taken from [2–4, 9–11]

Definition 1 A function $g : \mathbb{T} \to \mathbb{R}$ is considered ld-continuous if it is continuous at every left-dense point and the right-sided limit exists at every right-dense point. The set of all ld-continuous functions is denoted as $C_{ld}(\mathbb{T}, \mathbb{R})$. Note that, $g, h \in C_{ld}(\mathbb{T}, \mathbb{R})$, then $g \oplus_{\nu} h = g + h - \nu g h$ and $\ominus_{\nu} g = \frac{-g}{1-\nu g}$.

Definition 2 An ld-continuous function $g : \mathbb{T} \to \mathbb{R}$ is called complex ν-regressive (positively ν-regressive), if $1 - \nu(t)\, g(t) \neq 0$ $(1 - \nu g(t) \geq 0)$ for all $t \in \mathbb{T}_k$. The set of all ν-regressive functions is denoted by $\mathcal{R}_{\nu}(\mathbb{T}, \mathbb{R})$ and the set of all positively ν-regressive functions is denoted by $\mathcal{R}^+_{\nu}(\mathbb{T}, \mathbb{R})$ If time scale $\mathbb{T}$ has a right scattered minimum m, the $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$.

For $h > 0$, we have $\mathbb{C}_h(z) = \left\{z \in \mathbb{C} : z \neq \frac{1}{h}\right\}$ and $\mathbb{Z}_h = \left\{z \in \mathbb{C} : \frac{-\pi}{h} < Im(z) \leq \frac{\pi}{h}\right\}$ with $\mathbb{C}_0 = \mathbb{Z}_0 = \mathbb{C}$. Further, the Hilger real part of the complex number is given by $Re_h(z) = \frac{(1-|1-hz|)}{h}$. and imaginary part of complex numbers is given by $Im_h(z) = \frac{Arg(1-hz)}{h}$, where Arg denotes principal argument of a complex number. Note that, $Re_0(z) = Re(z)$ and $Im_0(z) = Im(z)$.

Definition 3 (Regulated Function) A function $\varphi : \mathbb{T} \to \mathbb{R}$ is said to be regulated if its right-sided limits exist(finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist(finite) at all left-dense points in $\mathbb{T}$.

Definition 4 (Exponential Function) If g in $\mathcal{R}^{\nu}_{c}(\mathbb{T}, \mathbb{R})$, then nabla exponential function is given by $e_g(t, t_0) = \exp\left(\int_{t_0}^{t} \xi_{\nu(t)} g(t)\, \nabla t\right)$, for $t, t_0 \in \mathbb{T}$, where the ν-cylindrical transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ is given by $\xi_h(z) = \frac{-1}{h} Log(1 - zh)$.

Lemma 1 *If $g \in \mathcal{R}^{\nu}_{c}(\mathbb{T}, \mathbb{C})$, then* $e^{\rho}_{\ominus g}(t, t_0) = \frac{e_{\ominus g}(t,t_0)}{1-\nu(t)g(t)}$ *and* $e^{\nabla}_{\ominus g}(t, t_0) = \ominus g\, e_{\ominus g}(t, t_0) = \frac{-g}{1-\nu g} e_{\ominus g}(t, t_0)$.

Definition 5 A function g belongs to the space $\mathcal{A}(\mathbb{T})$, if

i) g is piece-wise ld-continouus in every interval $[t_0, \tau] \cap \mathbb{T}$.

ii) g has exponential order k ,($k \in \mathcal{R}^{\nu}_{c}[t_0, \infty)$) on $[t_0, \infty)$ there exist $M > 0$, such that $|g(t)| \le M e_k(t, t_0)$ for all $t \in [t_0, \infty)_{\mathbb{T}}$.

The minimal-graininess function $\nu_* : \mathbb{T} \to \mathbb{R}^+$ is given as $\nu_*(t_0) = \inf \nu(t)$ for $t \in [t_0, \infty)_{\mathbb{T}}$.

Theorem 2 (Decay of the nabla exponetial function) *For* $\sup \mathbb{T} = \infty$, *let* $t_0 \in \mathbb{T}$ *and* $\omega \in \mathcal{R}^{\nu}_{c}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$. *Then for any* $z \in \mathbb{C}_{\nu_*(t_0)}(\omega)$, *we have the following properties,*

i) $|\hat{e}_{\omega \ominus z}(t, t_0)| \le \hat{e}_{\omega \ominus Re_{\nu_*(t_0)}(z)}(t, t_0)$ *for all* $\in [t_0, \infty)_{\mathbb{T}}$.

ii) $\lim_{t\to\infty} \hat{e}_{\omega \ominus Re_{\nu_*(t_0)}(z)}(t, t_0) = 0$.

iii) $\lim_{t\to\infty} \hat{e}_{\omega \ominus z}(t, t_0) = 0$.

Theorem 3 *Let* $a, b, c \in \mathbb{T}$, $\beta \in \mathbb{R}$ *and* $f, g \in C_{ld}$. *Then, the following statements hold.*

i) $\int_a^b [f(t) + g(t)]\, \nabla t = \int_a^b f(t)\, \nabla t + \int_a^b g(t)\, \nabla t;$

ii) $\int_a^b (\beta f(t))\, \nabla t = \beta \int_a^b f(t)\, \nabla t;$

iii) $\int_a^b f(t)\, \nabla t = -\int_b^a f(t)\, \nabla t;$

iv) $\int_a^b f(t)\, \nabla t = \int_a^c f(t)\, \nabla t + \int_c^b f(t)\, \nabla t;$

v) $\int_a^b f(\rho(t))\, g^{\nabla}(t)\, \nabla t = (fg)(b) - (fg)(a) - \int_a^b f^{\nabla}(t)\, g(t)\, \nabla t;$

vi) $\int_a^b f(t)\, g^{\nabla}(t)\, \nabla t = (fg)(b) - (fg)(a) - \int_a^b f^{\nabla}(t)\, g(\rho(t))\, \nabla t;$

vii) $\int_a^a f(t)\, \nabla t = 0;$

viii) *If* $|f(t)| \le g(t)$ *on* $[a, b)$, *then* $\left|\int_a^b f(t)\, \nabla t\right| \le \int_a^b g(t)\, \nabla t;$

ix) *If* $f(t) \ge 0$ *for all* $a \le t < b$, *then* $\int_a^b f(t)\, \nabla t \ge 0$.

Definition 6 The function $g(t_1, t_2) : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{C}$ is said to be of exponential type I, if there exists constants $M, K_1, K_2 > 0$ such that $|g(t_1, t_2)| \le M e^{K_1 t_1 + K_2 t_2}$ and g is said to be of exponential type II if there exists constants $M, K_1, K_2 > 0$ such that $|g(t_1, t_2)| \le M e_{K_1 \oplus K_2}(t_1, t_2, t_0, t_0')$.

1

3 Main Results

In this section, we extend the definition of the Laplace Elzaki transform on time scales introduced in [7].

Definition 7 Suppose that $\mathbb{T}_1$ and $\mathbb{T}_2$ are time scales such that $\sup\{\mathbb{T}_1, \mathbb{T}_2\} = \infty$ and $t_0 \in \mathbb{T}_1, t_0' \in \mathbb{T}_2$ are fixed, we define the α-Laplace Elzaki transform of an ld-continuous function $g : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{C}$ as

$$\mathcal{L}_{t_1}^{\alpha}\mathrm{E}_{t_2}[g(t_1, t_2)] = \mathrm{G}(z^{\alpha}, q) = q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right) g(t_1, t_2)\,\nabla t_1\nabla t_2,$$

for all $z^{\alpha}, \frac{1}{q} \in \mathrm{D}_{\nu}\{g\}$, where $\mathrm{D}_{\nu}\{g\}$ consists of all $z^{\alpha}, \frac{1}{q} \in \mathcal{R}_{\nu}(\mathbb{T}, \mathbb{R})$ for which the improper integral exists.

Theorem 4 [**ExistenceTheorem**] *If $g : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{C}$ is an ld-continuous function of exponential type II. Then the α- Laplace Elzaki transform of $g(t_1, t_2)$ exists for all regressive z and $\frac{1}{q}$ provided, $\mathrm{Re}_{\nu_1}(z) > \mathrm{K}_1$, $\mathrm{Re}_{\nu_2}\left(\frac{1}{q}\right) > \mathrm{K}_2$ and converges absolutely.*

Proof We have, $\mathcal{L}_{t_1}^{\alpha}\mathrm{E}_{t_2}[g(t_1, t_2)]$

$$= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right) g(t_1, t_2)\,\nabla t_1\nabla t_2$$

$$\leq q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} \left|e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right)\right| |g(t_1, t_2)|\,\nabla t_1\nabla t_2$$

$$\leq q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right) |g(t_1, t_2)|\,\nabla t_1\nabla t_2$$

$$\leq q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right) \mathrm{M}\hat{e}_{\mathrm{K}_1\oplus\mathrm{K}_2}\left(t_1, t_2, t_0, t_0'\right)\nabla t_1\nabla t_2$$

$$= \mathrm{M}q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} \frac{e_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right)}{(1-\nu_1(t)z^{\alpha})\left(1-\nu_2(t)\frac{1}{q}\right)}\hat{e}_{\mathrm{K}_1\oplus\mathrm{K}_2}\left(t_1, t_2, t_0, t_0'\right)\nabla t_1\nabla t_2$$

$$= \mathrm{M}\int_{t_0}^{\infty} \frac{\hat{e}_{\mathrm{K}_1\ominus z^{\alpha}}\left(t_1, t_2, t_0, t_0'\right)}{(1-\nu_1(t)z^{\alpha})}\nabla t_1\, q\int_{t_0'}^{\infty}\frac{\hat{e}_{\mathrm{K}_2\ominus\frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right)}{\left(1-\nu_2(t)\frac{1}{q}\right)}\nabla t_2$$

$$= \left[\frac{\mathrm{M}}{\mathrm{K}_1-z^{\alpha}}\int_{t_0}^{\infty}\frac{\mathrm{K}_1-z^{\alpha}}{(1-\nu_1 z^{\alpha})}\hat{e}_{\mathrm{K}_1\ominus z^{\alpha}}(t_1, t_0)\nabla t_1\right]\left[\frac{q}{\mathrm{K}_2-\frac{1}{q}}\int_{t_0'}^{\infty}\frac{\mathrm{K}_2-\frac{1}{q}}{1-\nu_2\frac{1}{q}}\hat{e}_{\mathrm{K}_2\ominus\frac{1}{q}}\left(t_2, t_0'\right)\nabla t_2\right]$$

$$= \left[\frac{\mathrm{M}}{\mathrm{K}_1-z^{\alpha}}\int_{t_0}^{\infty}(\mathrm{K}_1\ominus z^{\alpha})\,\hat{e}_{\mathrm{K}_1\ominus z^{\alpha}}(t_1, t_0)\nabla t_1\right]\left[\frac{q}{\mathrm{K}_2-\frac{1}{q}}\int_{t_0'}^{\infty}\left(\mathrm{K}_2\ominus\frac{1}{q}\right)\hat{e}_{\mathrm{K}_2\ominus\frac{1}{q}}\left(t_2, t_0'\right)\nabla t_2\right]$$

$$= \left[\frac{\mathrm{M}}{\mathrm{K}_1-z^{\alpha}}\int_{t_0}^{\infty} e^{\nabla}_{\mathrm{K}_1\ominus z^{\alpha}}(t_1, t_0)\nabla t_1\right]\left[\frac{q}{\mathrm{K}_2-\frac{1}{q}}\int_{t_0'}^{\infty} e^{\nabla}_{\mathrm{K}_2\ominus\frac{1}{q}}\left(t_2, t_0'\right)\nabla t_2\right]$$

$$= \frac{\mathrm{M}}{z^{\alpha}-\mathrm{K}_1}\;\frac{q^2}{1-q\mathrm{K}_2}$$

provided, $\lim_{t_1\to\infty} e_{\mathrm{K}_1\ominus z^{\alpha}}(t_1, t_0) \to 0$ and $\lim_{t_2\to\infty} e_{\mathrm{K}_2\ominus\frac{1}{q}}\left(t_2, t_0'\right) \to 0$ with $\mathrm{Re}_{\nu_1}(z) > \mathrm{K}_1$, $\mathrm{Re}_{\nu_2}\left(\frac{1}{q}\right) > \mathrm{K}_2$. □

Theorem 5 [**LinearityProperty**]
If c_1g_1 and c_2g_2 are the functions of class $\mathrm{A}(\mathbb{T})$ *with constants $c_1, c_2 \in \mathbb{R}$, then*
$\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[c_1g_1(t_1,t_2)+c_2g_2(t_1,t_2)] = c_1\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g_1(t_1,t_2)] + c_2\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g_2(t_1,t_2)]$.

Proof Let $c_1g_1(t_1,t_2)$ and $c_2g_2(t_1,t_2)$ be functions of class $\mathrm{A}(\mathbb{T})$ with exponential order k_1 and k_2 respectively. Then $\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g_1(t_1,t_2)]$ exists for all $z \in \mathbb{C}_{(v_*(t_0))}(k_1)$ with $\mathrm{Re}_{v_1}(z) > k_1$,and $\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g_2(t_1,t_2)]$exists for all $\frac{1}{q} \in \mathbb{C}_{(v_*(t_0))}(k_2)$ with $\mathrm{Re}_{v_2}\left(\frac{1}{q}\right) > k_2$. Then $\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[c_1g_1(t_1,t_2)+c_2g_2(t_1,t_2)]$ exists for all $z^{\alpha}, \frac{1}{q} \in \mathrm{D}_v\{g\}$ with $\mathrm{Re}_{v_1}(z), \mathrm{Re}_{v_2}\left(\frac{1}{q}\right) > max\{k_1,k_2\}$.

$$\begin{aligned}
\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[c_1g_1(t_1,t_2)+c_2g_2(t_1,t_2)] &= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1,t_2,t_0,t_0'\right)[c_1g_1(t_1,t_2)+c_2g_2(t_1,t_2)]\nabla t_1\nabla t_2\\
&= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1,t_2,t_0,t_0'\right)[c_1g_1(t_1,t_2)]\nabla t_1\nabla t_2\\
&+ q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1,t_2,t_0,t_0'\right)[c_2g_2(t_1,t_2)]\nabla t_1\nabla t_2\\
&= c_1\left[q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1,t_2,t_0,t_0'\right)[g_1(t_1,t_2)]\nabla t_1\nabla t_2\right]\\
&+ c_2\left[q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1,t_2,t_0,t_0'\right)[g_2(t_1,t_2)]\nabla t_1\nabla t_2\right]\\
&= c_1\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g_1(t_1,t_2)] + c_2\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g_2(t_1,t_2)].
\end{aligned}$$

□

3.1 The α-Laplace Elzaki transform of some elementary functions

We apply Definition 7 to find the α-Laplace Elzaki transform of some elementary functions as follows:

A) **g (t₁, t₂) = 1**
Proof:

$$\begin{aligned}
\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[1] &= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1,t_2,t_0,t_0'\right)\nabla t_1\nabla t_2\\
&= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} \frac{\hat{e}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1,t_0,t_2,t_0'\right)}{(1-v_1(t)z^{\alpha})\left(1-v_2(t)\frac{1}{q}\right)}\nabla t_1\nabla t_2\\
&= \int_{t_0}^{\infty}\frac{\hat{e}_{\ominus z^{\alpha}}(t_1,t_0)}{(1-v_1(t)z^{\alpha})}\nabla t_1\left(q\int_{t_0'}^{\infty}\frac{\hat{e}_{\ominus\frac{1}{q}}\left(t_2,t_0'\right)}{1-v_2(t)\frac{1}{q}}\nabla t_2\right)
\end{aligned}$$

$$= \frac{1}{-z^{\alpha}}\int_{t_0}^{\infty}\frac{(-z^{\alpha})\,\hat{e}_{\ominus z^{\alpha}}(t_1,t_0)}{(1-v_1(t)\,z^{\alpha})}\nabla t_1\left[\frac{q}{\frac{-1}{q}}\int_{t_0'}^{\infty}\frac{\frac{-1}{q}\,\hat{e}_{\ominus\frac{1}{q}}(t_2,t_0')}{1-v_2(t)\,\frac{1}{q}}\nabla t_2\right]$$

$$= \frac{-1}{(z^{\alpha})}\int_{t_0}^{\infty}(\ominus z^{\alpha})\,\hat{e}_{\ominus z^{\alpha}}(t_1,t_0)\,\nabla t_1\left[\frac{q}{\frac{-1}{q}}\int_{t_0'}^{\infty}\ominus\frac{1}{q}\,\hat{e}_{\ominus\frac{1}{q}}(t_2,t_0')\,\nabla t_2\right]$$

$$= \frac{-1}{(z^{\alpha})}\int_{t_0}^{\infty}e^{\nabla_1}_{\ominus z^{\alpha}}(t_1,t_0)\,\nabla t_1\left[-q^2\int_{t_0'}^{\infty}e^{\nabla_2}_{\ominus\frac{1}{q}}(t_2,t_0')\,\nabla t_2\right]$$

$$L^{\alpha}_{t_1}E_{t_2}[1] = \frac{q^2}{z^{\alpha}}$$

Provided, $\lim_{t_1\to\infty} e_{\ominus z^{\alpha}}(t_1,t_0)\to 0$ and $\lim_{t_2\to\infty} e_{\ominus\frac{1}{q}}(t_2,t_0')\to 0$ □

B) $\mathbf{g(t_1,t_2) = e_{a\oplus b}(t_1,t_2,t_0,t_0')}$

Proof:

$$L^{\alpha}_{t_1}E_{t_2}\left[e_{a\oplus b}(t_1,t_2,t_0,t_0')\right]$$

$$= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty}e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}(t_1,t_2,t_0,t_0')\,e_{a\oplus b}(t_1,t_2,t_0,t_0')\,\nabla t_1\nabla t_2$$

$$= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty}\frac{\hat{e}_{\ominus z^{\alpha}\ominus\frac{1}{q}}(t_1,t_0,t_2,t_0')\,e_{a\oplus b}(t_1,t_2,t_0,t_0')}{(1-v_1(t)z^{\alpha})\left(1-v_2(t)\frac{1}{q}\right)}\nabla t_1\nabla t_2$$

$$= \left[\int_{t_0}^{\infty}\frac{\hat{e}_{a\ominus z^{\alpha}}(t_1,t_0)}{(1-v_1(t)z^{\alpha})}\nabla t_1\right]\left[q\int_{t_0'}^{\infty}\frac{e_{b\ominus\frac{1}{q}}(t_2,t_0')}{\left(1-v_2(t)\frac{1}{q}\right)}\nabla t_2\right]$$

$$= \left[\frac{1}{(a-z^{\alpha})}\int_{t_0}^{\infty}\frac{(a-z^{\alpha})\hat{e}_{a\ominus z^{\alpha}}(t_1,t_0)}{(1-v_1(t)z^{\alpha})}\nabla t_1\right]\left[\frac{q}{\left(b-\frac{1}{q}\right)}\int_{t_0'}^{\infty}\frac{\left(b-\frac{1}{q}\right)e_{b\ominus\frac{1}{q}}(t_2,t_0)}{\left(1-v_2(t)\frac{1}{q}\right)}\nabla t_2\right]$$

$$= \left[\frac{1}{(a-z^{\alpha})}\int_{t_0}^{\infty}(a\ominus z^{\alpha})\,\hat{e}_{a\ominus z^{\alpha}}(t_1,t_0)\,\nabla t_1\right]\left[\frac{q}{\left(b-\frac{1}{q}\right)}\int_{t_0'}^{\infty}b\ominus\frac{1}{q}\,\hat{e}_{b\ominus\frac{1}{q}}(t_2,t_0')\,\nabla t_2\right]$$

$$= \left[\frac{1}{a-z^{\alpha}}\int_{t_0}^{\infty}e^{\nabla_1}_{a\ominus z^{\alpha}}(t_1,t_0)\,\nabla t_1\right]\left[\frac{q^2}{(bq-1)}\int_{t_0'}^{\infty}e^{\nabla_2}_{b\ominus\frac{1}{q}}(t_2,t_0')\,\nabla t_2\right]$$

$$= \left[\frac{1}{z^{\alpha}-a}\right]\left[\frac{q^2}{1-bq}\right]$$

$$L^{\alpha}_{t_1}E_{t_2}\left[e_{a\oplus b}(t_1,t_2,t_0,t_0')\right] = \frac{q^2}{(z^{\alpha}-a)(1-bq)}$$ □

C) $\mathbf{g(t_1,t_2) = \sin_{a\oplus b}(t_1,t_2,t_0,t_0')}$

Proof:

$$L^{\alpha}_{t_1}E_{t_2}\left[\sin_{a\oplus b}(t_1,t_2,t_0,t_0')\right]$$

$$= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty}e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}(t_1,t_2,t_0,t_0')\,\sin_{a\oplus b}(t_1,t_2,t_0,t_0')\,\nabla t_1\nabla t_2$$

$$= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty}e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}(t_1,t_2,t_0,t_0')\,\frac{e_{i(a\oplus b)}(t_1,t_2,t_0,t_0')-e_{\ominus i(a\oplus b)}(t_1,t_2,t_0,t_0')}{2i}\nabla t_1\nabla t_2$$

$$= L^{\alpha}_{t_1}E_{t_2}\left[\frac{e_{i(a\oplus b)}(t_1,t_2,t_0,t_0')-e_{\ominus i(a\oplus b)}(t_1,t_2,t_0,t_0')}{2i}\right]$$

$$= \frac{1}{2i}\left[\frac{q^2}{(z^\alpha-ai)(1-ibq)} - \frac{q^2}{(z^\alpha+ai)(1+ibq)}\right]$$

$$\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}\left[sin_{a\oplus b}\left(t_1, t_2, t_0, t_0'\right)\right] = \frac{q^2(z^\alpha bq+a)}{(z^{2\alpha}+a^2)(1+b^2q^2)}$$ □

D) **g** $(\mathbf{t_1}, \mathbf{t_2}) = \mathbf{cos}_{\mathbf{a\oplus b}}\left(\mathbf{t_1}, \mathbf{t_2}, \mathbf{t_0}, \mathbf{t_0'}\right)$

Proof:

$$\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}\left[cos_{a\oplus b}\left(t_1, t_2, t_0, t_0'\right)\right]$$

$$= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^\alpha \ominus \frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right) cos_{a\oplus b}\left(t_1, t_2, t_0, t_0'\right) \nabla t_1\nabla t_2$$

$$= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^\alpha \ominus \frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right) \left(\frac{e_{i(a\oplus b)}\left(t_1,t_2,t_0,t_0'\right) + e_{\ominus i(a\oplus b)}\left(t_1,t_2,t_0,t_0'\right)}{2}\right) \nabla t_1\nabla t_2$$

$$= \mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}\left[\frac{e_{i(a\oplus b)}\left(t_1,t_2,t_0,t_0'\right) + e_{\ominus i(a\oplus b)}\left(t_1,t_2,t_0,t_0'\right)}{2}\right]$$

$$= \frac{1}{2}\left[\frac{q^2}{(z^\alpha-ai)(1-ibq)} + \frac{q^2}{(z^\alpha+ai)(1+ibq)}\right]$$

$$\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}\left[cos_{a\oplus b}\left(t_1, t_2, t_0, t_0'\right)\right] = \frac{q^2(z^\alpha-abq)}{(z^{2\alpha}+a^2)(1+b^2q^2)}$$ □

E) **g** $(\mathbf{t_1}, \mathbf{t_2}) = \mathbf{sinh}_{\mathbf{a\oplus b}}\left(\mathbf{t_1}, \mathbf{t_2}, \mathbf{t_0}, \mathbf{t_0'}\right)$

Proof:

$$\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}\left[sinh_{a\oplus b}\left(t_1, t_2, t_0, t_0'\right)\right]$$

$$= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^\alpha \ominus \frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right) sinh_{a\oplus b}\left(t_1, t_2, t_0, t_0'\right) \nabla t_1\nabla t_2$$

$$= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^\alpha \ominus \frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right) \left(\frac{e_{(a\oplus b)}\left(t_1,t_2,t_0,t_0'\right) - e_{\ominus(a\oplus b)}\left(t_1,t_2,t_0,t_0'\right)}{2}\right) \nabla t_1\nabla t_2$$

$$= \mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}\left[\frac{e_{(a\oplus b)}\left(t_1,t_2,t_0,t_0'\right) - e_{\ominus(a\oplus b)}\left(t_1,t_2,t_0,t_0'\right)}{2}\right]$$

$$= \frac{1}{2}\left[\frac{q^2}{(z^\alpha-a)(1-bq)} - \frac{q^2}{(z^\alpha+a)(1+bq)}\right]$$

$$\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}\left[sinh_{a\oplus b}\left(t_1, t_2, t_0, t_0'\right)\right] = \frac{q^2(z^\alpha bq+a)}{(z^{2\alpha}-a^2)(1-b^2q^2)}$$ □

F) **g** $(\mathbf{t_1}, \mathbf{t_2}) = \mathbf{cosh}_{\mathbf{a\oplus b}}\left(\mathbf{t_1}, \mathbf{t_2}, \mathbf{t_0}, \mathbf{t_0'}\right)$

Proof:

$$\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}\left[cosh_{a\oplus b}\left(t_1, t_2, t_0, t_0'\right)\right]$$

$$= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^\alpha \ominus \frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right) cosh_{a\oplus b}\left(t_1, t_2, t_0, t_0'\right) \nabla t_1\nabla t_2$$

$$= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^\alpha \ominus \frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right) \left(\frac{e_{(a\oplus b)}\left(t_1,t_2,t_0,t_0'\right) + e_{\ominus(a\oplus b)}\left(t_1,t_2,t_0,t_0'\right)}{2}\right) \nabla t_1\nabla t_2$$

$$= \mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}\left[\frac{e_{(a\oplus b)}\left(t_1,t_2,t_0,t_0'\right) + e_{\ominus(a\oplus b)}\left(t_1,t_2,t_0,t_0'\right)}{2}\right]$$

$$= \frac{1}{2}\left[\frac{q^2}{(z^\alpha-a)(1-bq)} + \frac{q^2}{(z^\alpha+a)(1+bq)}\right]$$

$$\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}\left[cosh_{a\oplus b}\left(t_1, t_2, t_0, t_0'\right)\right] = \frac{q^2(z^\alpha+abq)}{(z^{2\alpha}-a^2)(1-b^2q^2)}$$ □

G) **g** $(\mathbf{t_1}, \mathbf{t_2}) = \mathbf{t_1 t_2}$

Proof:

$$\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}\left[t_1 t_2\right]$$

$$= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^\alpha \ominus \frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right) t_1 t_2 \nabla t_1 \nabla t_2$$

$$= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} \frac{e_{\ominus z^\alpha \ominus \frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right)}{(1-v_1(t)z^\alpha)\left(1-v_2(t)\frac{1}{q}\right)} t_1 t_2 \nabla t_1 \nabla t_2$$

$$= \left[\int_{t_0}^{\infty} \frac{e_{\ominus z^\alpha}(t_1, t_0) t_1}{(1-v_1(t)z^\alpha)}\nabla t_1\right]\left[q\int_{t_0'}^{\infty} \frac{e_{\ominus \frac{1}{q}}\left(t_2, t_0'\right) t_2}{\left(1-v_2(t)\frac{1}{q}\right)}\nabla t_2\right]$$

$$= \left[\frac{1}{(-z^\alpha)}\int_{t_0}^{\infty} \frac{(-z^\alpha) e_{\ominus z^\alpha}(t_1, t_0) t_1}{(1-v_1(t)z^\alpha)}\nabla t_1\right]\left[\frac{q}{\left(\frac{-1}{q}\right)}\int_{t_0}^{\infty} \frac{\left(\frac{-1}{q}\right) e_{\ominus \frac{1}{q}}\left(t_2, t_0'\right) t_2}{\left(1-v_2(t)\frac{1}{q}\right)}\nabla t_2\right]$$

$$= \left[\frac{1}{(-z^\alpha)}\int_{t_0}^{\infty} (\ominus z^\alpha)\, e_{\ominus z^\alpha}(t_1, t_0)\, t_1 \nabla t_1\right]\left[\frac{q}{\left(\frac{-1}{q}\right)}\int_{t_0'}^{\infty} \ominus\frac{1}{q}\; e_{\ominus \frac{1}{q}}\left(t_2, t_0'\right) t_2 \nabla t_2\right]$$

$$= \left[\frac{1}{(-z^\alpha)}\int_{t_0}^{\infty} e^{\nabla_1}_{\ominus z^\alpha}(t_1, t_0)\, t_1 \nabla t_1\right]\left[\frac{q}{\left(\frac{-1}{q}\right)}\int_{t_0}^{\infty} e^{\nabla_2}_{\ominus \frac{1}{q}}\left(t_2, t_0'\right) t_2 \nabla t_2\right]$$

Applying the integration by parts rule, we get

$$= \frac{q^3}{z^{2\alpha}}$$

In general,

$$\mathrm{L}^\alpha_{t_1}\mathrm{E}_{t_2}\left[g_m(t_1, t_0)\, g_n\left(t_2, t_0'\right)\right] = \mathrm{L}^\alpha_{t_1}\mathrm{E}_{t_2}\left[t_1^m t_2^n\right] = \frac{m!n!q^{n+2}}{z^{(m+1)\alpha}} \qquad \square$$

Theorem 6 (Basic Derivative properties of α-Laplace Elzaki Transform) *Let $g(t_1, t_2) : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{C}$ be an ld-continuous function such that $g^{\nabla_1}(t_1, t_2) = \frac{\partial g(t_1,t_2)}{\nabla_1 t_1}$, $g^{\nabla_1^2}(t_1, t_2) = \frac{\partial^2 g(t_1,t_2)}{\nabla_1 t_1^2}$, $g^{\nabla_2}(t_1, t_2) = \frac{\partial g(t_1,t_2)}{\nabla_2 t_2}$, $g^{\nabla_2^2}(t_1, t_2) = \frac{\partial^2 g(t_1,t_2)}{\nabla_2 t_2^2}$, are also ld-continuous. Then*

1) $\mathrm{L}^\alpha_{t_1}\mathrm{E}_{t_2}\left[g^{\nabla_1}(t_1, t_2)\right] = z^\alpha \mathrm{L}^\alpha_{t_1}\mathrm{E}_{t_2}\left[g(t_1, t_2)\right] - \mathrm{E}_{t_2}\left[g(t_0, t_2)\right]$,
2) $\mathrm{L}^\alpha_{t_1}\mathrm{E}_{t_2}\left[g^{\nabla_2}(t_1, t_2)\right] = \frac{1}{q}\mathrm{L}^\alpha_{t_1}\mathrm{E}_{t_2}\left[g(t_1, t_2)\right] - q\,\mathrm{L}^\alpha_{t_1}\left[g\left(t_1, t_0'\right)\right]$,
3) $\mathrm{L}^\alpha_{t_1}\mathrm{E}_{t_2}\left[g^{\nabla_1^2}(t_1, t_2)\right] = z^{2\alpha}\,\mathrm{L}^\alpha_{t_1}\mathrm{E}_{t_2}\left[g(t_1, t_2)\right] - z^\alpha\,\mathrm{E}_{t_2}\left[g(t_0, t_2)\right] - \mathrm{E}_{t_2}\left[g^{\nabla_1}(t_0, t_2)\right]$,
4) $\mathrm{L}^\alpha_{t_1}\mathrm{E}_{t_2}\left[g^{\nabla_2^2}(t_1, t_2)\right] = \frac{1}{q^2}\mathrm{L}^\alpha_{t_1}\mathrm{E}_{t_2}\left[g(t_1, t_2)\right] - q\,\mathrm{L}^\alpha_{t_1}\left[g^{\nabla_2}\left(t_1, t_0'\right)\right] - \mathrm{L}^\alpha_{t_1}\left[g\left(t_1, t_0'\right)\right]$.

Proof 1) $\mathrm{L}^\alpha_{t_1}\mathrm{E}_{t_2}\left[g^{\nabla_1}(t_1, t_2)\right] = z^\alpha \mathrm{L}^\alpha_{t_1}\mathrm{E}_{t_2}\left[g(t_1, t_2)\right] - \mathrm{E}_{t_2}\left[g(t_0, t_2)\right]$

$$\mathrm{L}^\alpha_{t_1}\mathrm{E}_{t_2}\left[g^{\nabla_1}(t_1, t_2)\right]$$

$$= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^\alpha \ominus \frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right) g^{\nabla_1}(t_1, t_2)\,\nabla t_1 \nabla t_2$$

$$= q\int_{t_0}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right)\left[\int_{t_0'}^{\infty} e^{\rho_1}_{\ominus z^\alpha}(t_1, t_0)\, g^{\nabla_1}(t_1, t_2)\,\nabla t_1\right]\nabla t_2$$

$$= q\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right)\left[\int_{t_0}^{\infty}\left[\left(e_{\ominus z^\alpha}(t_1, t_0)\, g(t_1, t_2)\right)^{\nabla_1} - e^{\nabla_1}_{\ominus z^\alpha}(t_1, t_0)\, g(t_1, t_2)\right]\nabla t_1\right]\nabla t_2$$

$$= q\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right)\left[-g(t_0, t_2) - \int_{t_0}^{\infty} e^{\nabla_1}_{\ominus z^\alpha}(t_1, t_0)\, g(t_1, t_2)\,\nabla t_1\right]\nabla t_2$$

$$= q\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right)\left[-g(t_0, t_2) - \int_{t_0}^{\infty} \ominus z^\alpha \hat{e}_{\ominus z^\alpha}(t_1, t_0)\, g(t_1, t_2))\nabla t_1\right]\nabla t_2$$

$$= q\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right)\left[-g(t_0, t_2) - \int_{t_0}^{\infty} \frac{(-z^\alpha)\hat{e}_{\ominus z^\alpha}(t_1, t_0)}{(1-v_1(t)z^\alpha)} g(t_1, t_2))\nabla t_1\right]\nabla t_2$$

$$= q\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right)\left[-g(t_0, t_2) + z^\alpha\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^\alpha}(t_1, t_0)\, g(t_1, t_2))\nabla t_1\right]\nabla t_2$$

$$= -q\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right) g\left(t_0, t_2\right)\nabla t_2 + z^{\alpha} q\int_{t_0}^{\infty} e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right) g\left(t_1, t_2\right)\nabla t_1\nabla t_2$$

$$= -\mathcal{E}_{t_2}\left[g\left(t_0, t_2\right)\right] + z^{\alpha}\mathcal{L}^{\alpha}_{t_1}\mathcal{E}_{t_2}\left[g\left(t_1, t_2\right)\right]$$

$$= z^{\alpha}\mathcal{L}^{\alpha}_{t_1}\mathcal{E}_{t_2}\left[g\left(t_1, t_2\right)\right] - \mathcal{E}_{t_2}\left[g\left(t_0, t_2\right)\right]. \qquad \square$$

2)
$$\mathcal{L}^{\alpha}_{t_1}\mathcal{E}_{t_2}\left[g^{\nabla_2}\left(t_1, t_2\right)\right] = \frac{1}{q}\mathcal{L}^{\alpha}_{t_1}\mathcal{E}_{t_2}\left[g\left(t_1, t_2\right)\right] - q\,\mathcal{L}^{\alpha}_{t_1}\left[g\left(t_1, t_0'\right)\right]$$

$$\mathcal{L}^{\alpha}_{t_1}\mathcal{E}_{t_2}\left[g^{\nabla_2}\left(t_1, t_2\right)\right]$$

$$= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right) g^{\nabla_2}\left(t_1, t_2\right)\nabla t_1\nabla t_2$$

$$= q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}\left(t_1, t_0\right)\left[\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right) g^{\nabla_2}\left(t_1, t_2\right)\nabla t_2\right]\nabla t_1$$

$$= q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}\left(t_1, t_0\right)\int_{t_0'}^{\infty}\left[\left(e_{\ominus\frac{1}{q}}\left(t_2, t_0'\right) g\left(t_1, t_2\right)\right)^{\nabla_2} - e^{\nabla_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right) g\left(t_1, t_2\right)\right]\nabla t_2\,\nabla t_1$$

$$= q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}\left(t_1, t_0\right)\left[-g\left(t_1, t_0'\right) - \int_{t_0'}^{\infty} e^{\nabla_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right) g\left(t_1, t_2\right)\nabla t_2\right]\nabla t_1$$

$$= q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}\left(t_1, t_0\right)\left[-g\left(t_1, t_0'\right) - \int_{t_0'}^{\infty}\ominus\frac{1}{q}\, e_{\ominus\frac{1}{q}}\left(t_2, t_0'\right) g\left(t_1, t_2\right)\nabla t_2\right]\nabla t_1$$

$$= q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}\left(t_1, t_0\right)\left[-g\left(t_1, t_0'\right) - \int_{t_0'}^{\infty}\frac{\left(-\frac{1}{q}\right)e_{\ominus\frac{1}{q}}\left(t_2, t_0'\right)}{\left(1-\nu_2(t)\frac{1}{q}\right)} g\left(t_1, t_2\right)\nabla t_2\right]\nabla t_1$$

$$= q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}\left(t_1, t_0\right)\left[-g\left(t_1, t_0'\right) + \frac{1}{q}\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right) g\left(t_1, t_2\right)\nabla t_2\right]\nabla t_1$$

$$= -q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}\left(t_1, t_0\right) g\left(t_1, t_0'\right)\nabla t_1 + \frac{1}{q}\left[q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right) g\left(t_1, t_2\right)\nabla t_1\nabla t_2\right]$$

$$= \frac{1}{q}\mathcal{L}^{\alpha}_{t_1}\mathcal{E}_{t_2}\left[g\left(t_1, t_2\right)\right] - q\,\mathcal{L}^{\alpha}_{t_1}\left[g\left(t_1, t_0'\right)\right] \qquad \square$$

3)
$$\mathcal{L}^{\alpha}_{t_1}\mathcal{E}_{t_2}\left[g^{\nabla_1^2}\left(t_1, t_2\right)\right] = z^{2\alpha}\mathcal{L}^{\alpha}_{t_1}\mathcal{E}_{t_2}\left[g\left(t_1, t_2\right)\right] - z^{\alpha}\mathcal{E}_{t_2}\left[g\left(t_0, t_2\right)\right] - \mathcal{E}_{t_2}\left[g^{\nabla_1}\left(t_0, t_2\right)\right]$$

$$\mathcal{L}^{\alpha}_{t_1}\mathcal{E}_{t_2}\left[g^{\nabla_1^2}\left(t_1, t_2\right)\right]$$

$$= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right) g^{\nabla_1^2}\left(t_1, t_2\right)\nabla t_1\nabla t_2$$

$$= q\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right)\left[\int_{t_0}^{\infty}\left[\left(e_{\ominus z^{\alpha}}\left(t_1, t_0\right) g^{\nabla_1}\left(t_1, t_2\right)\right)^{\nabla_1} - e^{\nabla_1}_{\ominus z^{\alpha}}\left(t_1, t_0\right) g^{\nabla_1}\left(t_1, t_2\right)\right]\nabla t_1\right]\nabla t_2$$

$$= q\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right)\left[-g^{\nabla_1}\left(t_0, t_2\right) + z^{\alpha}\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}\left(t_1, t_0\right) g^{\nabla_1}\left(t_1, t_2\right)\nabla t_1\right]\nabla t_2$$

$$= q\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right)$$

$$\times\left[-g^{\nabla_1}\left(t_0, t_2\right) + z^{\alpha}\int_{t_0}^{\infty}\left[\left(e_{\ominus z^{\alpha}}\left(t_1, t_0\right) g\left(t_1, t_2\right)\right)^{\nabla_1} - e^{\nabla_1}_{\ominus z^{\alpha}}\left(t_1, t_0\right) g\left(t_1, t_2\right)\right]\nabla t_1\right]\nabla t_2$$

$$= -q\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right) g^{\nabla_1}\left(t_0, t_2\right)\nabla t_2 +$$

$$z^{\alpha} q\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right)\left[-g\left(t_0, t_2\right) - \int_{t_0}^{\infty} e^{\nabla_1}_{\ominus z^{\alpha}}\left(t_1, t_0\right) g\left(t_1, t_2\right)\nabla t_1\right]\nabla t_2$$

$$= -q\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right) g^{\nabla_1}\left(t_0, t_2\right)\nabla t_2 +$$

$$z^{\alpha} q\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right)\left[-g\left(t_0, t_2\right) - \int_{t_0}^{\infty}\left(\ominus z^{\alpha}\right)\left(e_{\ominus z^{\alpha}}\left(t_1, t_0\right) g\left(t_1, t_2\right)\right)\nabla t_1\right]\nabla t_2$$

$$= -q\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right) g^{\nabla_1}\left(t_0, t_2\right)\nabla t_2 +$$

$$z^{\alpha} q\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right)\left[-g\left(t_0, t_2\right) - \int_{t_0}^{\infty}\frac{\left(-z^{\alpha}\right)e_{\ominus z^{\alpha}}\left(t_1, t_0\right)}{\left(1-\nu_1(t)z^{\alpha}\right)} g\left(t_1, t_2\right)\nabla t_1\right]\nabla t_2$$

$$= -q\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right) g^{\nabla_1}\left(t_0, t_2\right)\nabla t_2 +$$

$$z^{\alpha} q \int_{t_0'}^{\infty} e^{\rho_2}_{\ominus \frac{1}{q}}\left(t_2, t_0'\right)\left[-g(t_0, t_2) + z^{\alpha}\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}(t_1, t_0)\, g(t_1, t_2)\nabla t_1\right]\nabla t_2$$

$$= q\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus \frac{1}{q}}\left(t_2, t_0'\right) g^{\nabla_1}(t_0, t_2)\nabla t_2 - z^{\alpha} q\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus \frac{1}{q}}\left(t_2, t_0'\right) g(t_0, t_2)\nabla t_2$$

$$+ z^{2\alpha} q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right) g(t_1, t_2)\nabla t_1\nabla t_2$$

$$= z^{2\alpha}\, L^{\alpha}_{t_1} E_{t_2}\left[g(t_1, t_2)\right] - z^{\alpha}\, E_{t_2}\left[g(t_0, t_2)\right] - E_{t_2}\left[g^{\nabla_1}(t_0, t_2)\right]$$ □

4) $L^{\alpha}_{t_1}E_{t_2}\left[g^{\nabla_2^2}(t_1, t_2)\right] = \frac{1}{q^2}L^{\alpha}_{t_1}E_{t_2}\left[g(t_1, t_2)\right] - q\, L^{\alpha}_{t_1}\left[g^{\nabla_2}\left(t_1, t_0'\right)\right] - L^{\alpha}_{t_1}\left[g\left(t_1, t_0'\right)\right]$

Consider, $L^{\alpha}_{t_1}E_{t_2}\left[g^{\nabla_2^2}(t_1, t_2)\right]$

$$= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right) g^{\nabla_2^2}(t_1, t_2)\nabla t_1\nabla t_2$$

$$= q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}(t_1, t_0)\int_{t_0'}^{\infty}\left[\left(e_{\ominus\frac{1}{q}}\left(t_2, t_0'\right) g_{\nabla_2}(t_1, t_2)\right)^{\nabla_2} - e^{\nabla_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right) g_{\nabla_2}(t_1, t_2)\right]\nabla t_2\nabla t_1$$

$$= q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}(t_1, t_0)\left[-g^{\nabla_2}\left(t_1, t_0'\right) + \frac{1}{q}\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right) g^{\nabla_2}(t_1, t_2)\nabla t_2\right]\nabla t_1$$

$$= q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}(t_1, t_0)$$

$$\times\left[-g^{\nabla_2}\left(t_1, t_0'\right) + \frac{1}{q}\int_{t_0'}^{\infty}\left[\left(e_{\ominus\frac{1}{q}}\left(t_2, t_0'\right) g(t_1, t_2)\right)^{\nabla_2} - e^{\nabla_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right) g(t_1, t_2)\right]\nabla t_2\right]\nabla t_1$$

$$= -q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}(t_1, t_0)\, g^{\nabla_2}\left(t_1, t_0'\right)\nabla t_1 + \frac{1}{q}\left[q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}(t_1, t_0)\left[-g\left(t_1, t_0'\right) - \int_{t_0'}^{\infty} e^{\nabla_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right) g(t_1, t_2)\nabla t_2\right]\nabla t_1\right]$$

$$= -q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}(t_1, t_0)\, g^{\nabla_2}\left(t_1, t_0'\right)\nabla t_1 + \frac{1}{q}\left[q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}(t_1, t_0)\left[-g\left(t_1, t_0'\right) - \int_{t_0'}^{\infty} \ominus\frac{1}{q}\, e_{\ominus\frac{1}{q}}\left(t_2, t_0'\right) g(t_1, t_2)\nabla t_2\right]\nabla t_1\right]$$

$$= -q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}(t_1, t_0)\, g^{\nabla_2}\left(t_1, t_0'\right)\nabla t_1 + \frac{1}{q}\left[q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}(t_1, t_0)\left[-g\left(t_1, t_0'\right) - \int_{t_0'}^{\infty}\frac{\left(-\frac{1}{q}\right)e_{\ominus\frac{1}{q}}\left(t_2, t_0'\right)}{\left(1-\nu_2(t)\frac{1}{q}\right)} g(t_1, t_2)\nabla t_2\right]\nabla t_1\right]$$

$$= -q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}(t_1, t_0)\, g^{\nabla_2}\left(t_1, t_0'\right)\nabla t_1 + \frac{1}{q}\left[q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}(t_1, t_0)\left[-g\left(t_1, t_0'\right) + \frac{1}{q}\int_{t_0'}^{\infty} e^{\rho_2}_{\ominus\frac{1}{q}}\left(t_2, t_0'\right) g(t_1, t_2)\nabla t_2\right]\nabla t_1\right]$$

$$= -q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}(t_1, t_0)\, g^{\nabla_2}\left(t_1, t_0'\right)\nabla t_1 + \frac{1}{q}\left[q\int_{t_0}^{\infty} e^{\rho_1}_{\ominus z^{\alpha}}(t_1, t_0)\left[-g\left(t_1, t_0'\right)\right]\nabla t_1 + \frac{1}{q}\left[q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1, t_2, t_0, t_0'\right) g(t_1, t_2)\nabla t_1\nabla t_2\right]\right]$$

$$= -q\, L^{\alpha}_{t_1}\left[g^{\nabla_2}\left(t_1, t_0'\right)\right] - L^{\alpha}_{t_1}\left[g\left(t_1, t_0'\right)\right] + \frac{1}{q^2}L^{\alpha}_{t_1}E_{t_2}\left[g(t_1, t_2)\right]$$

$$= \frac{1}{q^2}L^{\alpha}_{t_1}E_{t_2}\left[g(t_1, t_2)\right] - q\, L^{\alpha}_{t_1}\left[g^{\nabla_2}\left(t_1, t_0'\right)\right] - L^{\alpha}_{t_1}\left[g\left(t_1, t_0'\right)\right].$$

□

Theorem 7 *If* $\beta_1 \in \mathbb{T}_1$ *and* $\beta_2 \in \mathbb{T}_2$ *with* $\beta_1, \beta_2 > 0$

$$H_{\beta_1,\beta_2}(t_1, t_2) = \begin{cases} 0 : & t_1 \in \mathbb{T}_1, t_2 \in \mathbb{T}_2 \text{ and } t_1 < \beta_1, t_2 < \beta_2 \\ 1 : & t_1 \in \mathbb{T}_1, t_2 \in \mathbb{T}_2 \text{ and } t_1 \geq \beta_1, t_2 \geq \beta_2 \end{cases}$$

Then $L^{\alpha}_{t_1}E_{t_2}\left[H_{\beta_1,\beta_2}(t_1, t_2)\right] = \frac{q^2}{z^{\alpha}} e_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(\beta_1, \beta_2, t_0, t_0'\right).$

Proof Given $\beta_1 \in \mathrm{T}_1$ and $\beta_2 \in \mathrm{T}_2$ with $\beta_1, \beta_2 > 0$

$$\mathrm{H}_{\beta_1,\beta_2}(t_1,t_2) = \begin{cases} 0: & t_1 \in \mathrm{T}_1, t_2 \in \mathrm{T}_2 \text{ and } t_1 < \beta_1, t_2 < \beta_2 \\ 1: & t_1 \in \mathrm{T}_1, t_2 \in \mathrm{T}_2 \text{ and } t_1 \geq \beta_1, t_2 \geq \beta_2 \end{cases}$$

Then consider,

$$\begin{aligned}
\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}\left[\mathrm{H}_{\beta_1,\beta_2}(t_1,t_2)\right] &= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1,t_2,t_0,t_0'\right)\mathrm{H}_{\beta_1,\beta_2}(t_1,t_2)\,\nabla t_1\nabla t_2\\
&= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} \frac{e_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1,t_2,t_0,t_0'\right)}{(1-v_1z^{\alpha})\left(1-v_2\frac{1}{q}\right)}\mathrm{H}_{\beta_1,\beta_2}(t_1,t_2)\,\nabla t_1\nabla t_2\\
&= q\int_{\beta_1}^{\infty}\int_{\beta_2}^{\infty} \frac{e_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1,t_2,t_0,t_0'\right)}{(1-v_1z^{\alpha})\left(1-v_2\frac{1}{q}\right)}\nabla t_1\nabla t_2\\
&= \left[\int_{\beta_1}^{\infty}\frac{e_{\ominus z^{\alpha}}(t_1,t_0)}{(1-v_1z^{\alpha})}\nabla t_1\right]\left[q\int_{\beta_2}^{\infty}\frac{e_{\ominus\frac{1}{q}}\left(t_2,t_0'\right)}{1-v_2\frac{1}{q}}\nabla t_2\right]\\
&= \frac{1}{(-z^{\alpha})}\int_{\beta_1}^{\infty}\frac{(-z^{\alpha})\,e_{\ominus z^{\alpha}}(t_1,t_0)}{(1-v_1z^{\alpha})}\nabla t_1\left[\frac{q}{\frac{-1}{q}}\int_{\beta_2}^{\infty}\frac{\frac{-1}{q}\,e_{\ominus\frac{1}{q}}\left(t_2,t_0'\right)}{1-v_2\frac{1}{q}}\nabla t_2\right]\\
&= \frac{1}{(-z^{\alpha})}\int_{\beta_1}^{\infty}\ominus z^{\alpha}\,e_{\ominus z^{\alpha}}(t_1,t_0)\nabla t_1\left[\frac{q}{\frac{-1}{q}}\int_{\beta_2}^{\infty}\left(\ominus\frac{1}{q}\right)e_{\ominus\frac{1}{q}}\left(t_2,t_0'\right)\nabla t_2\right]\\
&= \frac{1}{(-z^{\alpha})}\int_{\beta_1}^{\infty}e^{\nabla_1}_{\ominus z^{\alpha}}(t_1,t_0)\left(-q^2\int_{\beta_2}^{\infty}e^{\nabla_2}_{\ominus\frac{1}{q}}\left(t_2,t_0'\right)\nabla t_2\right)\nabla t_1\\
&= \frac{1}{(z^{\alpha})}e_{\ominus z^{\alpha}}(\beta_1,t_0)\left[q^2 e_{\ominus\frac{1}{q}}\left(\beta_2,t_0'\right)\right]\\
&= \frac{q^2}{z^{\alpha}}e_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(\beta_1,\beta_2,t_0,t_0'\right).
\end{aligned}$$

□

Theorem 8 *If* $\mathrm{G}(z^{\alpha},q) = \mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)]$, *then*

$$\begin{aligned}
\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}\left[\mathrm{H}_{\beta_1,\beta_2}(t_1,t_2)\,g(t_1,t_2)\right] &= e_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(\beta_1,\beta_2,t_0,t_0'\right)\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)]\\
&= e_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(\beta_1,\beta_2,t_0,t_0'\right)\mathrm{G}\left(z^{\alpha},q\right).
\end{aligned}$$

Proof Given $\mathrm{G}(z^{\alpha},q) = \mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)]$.

$$\begin{aligned}
\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}\left[\mathrm{H}_{\beta_1,\beta_2}(t_1,t_2)\,g(t_1,t_2)\right] &= = q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1,\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1,t_2,t_0,t_0'\right)\mathrm{H}_{\beta_1,\beta_2}(t_1,t_2)\,g(t_1,t_2)\,\nabla t_1\nabla t_2\\
&= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty}\frac{e_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1,t_2,t_0,t_0'\right)}{(1-v_1z^{\alpha})\left(1-v_2\frac{1}{q}\right)}\mathrm{H}_{\beta_1,\beta_2}(t_1,t_2)\,g(t_1,t_2)\,\nabla t_1\nabla t_2
\end{aligned}$$

$$= q\int_{\beta_1}^{\infty}\int_{\beta_2}^{\infty}\frac{e_{\ominus z^{\alpha}\ominus\frac{1}{q}}(t_1,t_2,\beta_1,\beta_2)}{(1-v_1z^{\alpha})\left(1-v_2\frac{1}{q}\right)}e_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(\beta_1,\beta_2,t_0,t_0'\right)g(t_1,t_2)\nabla t_1\nabla t_2$$

$$= e_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(\beta_1,\beta_2,t_0,t_0'\right)q\int_{\beta_1}^{\infty}\int_{\beta_2}^{\infty}\frac{e_{\ominus z^{\alpha}\ominus\frac{1}{q}}(t_1,t_2,\beta_1,\beta_2)}{(1-v_1z^{\alpha})\left(1-v_2\frac{1}{q}\right)}g(t_1,t_2)\nabla t_1\nabla t_2$$

$$= e_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(\beta_1,\beta_2,t_0,t_0'\right)\mathrm{L}_{t_1}^{\alpha}\mathrm{E}_{t_2}[g(t_1,t_2)]$$

$$= e_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(\beta_1,\beta_2,t_0,t_0'\right)\mathrm{G}\left(z^{\alpha},q\right).$$

□

Definition 8 [9, 10] Let $g_1 : \mathrm{T}_1 \times \mathrm{T}_2 \to \mathbb{C}$ and $g_2 : \mathrm{T}_1 \times \mathrm{T}_2 \to \mathbb{C}$ be ∇- integrable functions, then the double convolution of $g_1(t_1,t_2)$ and $g_2(t_1,t_2)$ is given by

$$(g_1 ** g_2)(t_1,t_2) = \int_{t_0}^{t_1}\int_{t_0'}^{t_2} g_1(t_1,t_2,\rho_1(u_1),\rho_2(u_2))\,g_2(u_1,u_2)\nabla u_1\nabla u_2.$$

Theorem 9 **(Convolution Theorem)** *Let* $g_1 : \mathrm{T}_1 \times \mathrm{T}_2 \to \mathbb{C}$ *and* $g_2 : \mathrm{T}_1 \times \mathrm{T}_2 \to \mathbb{C}$ *be ld-continous functions of exponential type II having the* $\alpha-$ *Laplace Elzaki transform* $\mathrm{L}_{t_1}^{\alpha}\mathrm{E}_{t_2}[g_1(t_1,t_2)]$ *and* $\mathrm{L}_{t_1}^{\alpha}\mathrm{E}_{t_2}[g_2(t_1,t_2)]$ *respectively. Then*
$\mathrm{L}_{t_1}^{\alpha}\mathrm{E}_{t_2}[(g_1 ** g_2)(t_1,t_2)] = \frac{1}{q}\mathrm{L}_{t_1}^{\alpha}\mathrm{E}_{t_2}[g_1(t_1,t_2)]\cdot\mathrm{L}_{t_1}^{\alpha}\mathrm{E}_{t_2}[g_2(t_1,t_2)].$

Proof $\mathrm{L}_{t_1}^{\alpha}\mathrm{E}_{t_2}[(g_1 ** g_2)(t_1,t_2)]$

$$= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1,\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1,t_2,t_0,t_0'\right)[(g_1 ** g_2)(t_1,t_2)]\nabla t_1\nabla t_2$$

$$= q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\rho_1,\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1,t_2,t_0,t_0'\right)\left[\int_{t_0}^{t_1}\int_{t_0'}^{t_2} g_1(t_1,t_2,\rho_1(u_1),\rho_2(u_2))\,g_2(u_1,u_2)\nabla u_1\nabla u_2\right]\nabla t_1\nabla t_2$$

$$= \int_{t_0}^{\infty}\int_{t_0'}^{\infty} g_2(u_1,u_2)\left(q\int_{\rho_1(u_1)}^{\infty}\int_{\rho_2(u_2)}^{\infty} g_1(t_1,t_2,\rho_1(u_1),\rho_2(u_2))\cdot e^{\rho_1,\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1,t_2,t_0,t_0'\right)\nabla t_1\nabla t_2\right)\nabla u_1\nabla u_2$$

$$= \int_{t_0}^{\infty}\int_{t_0'}^{\infty} g_2(u_1,u_2)$$

$$\times\left[q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} g_1(t_1,t_2,\rho_1(u_1),\rho_2(u_2))\cdot\mathrm{H}_{\rho_1(u_1),\rho_2(u_2)}(t_1,t_2)\cdot e^{\rho_1,\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(t_1,t_2,t_0,t_0'\right)\nabla t_1\nabla t_2\right]\nabla u_1\nabla u_2$$

$$= \int_{t_0}^{\infty}\int_{t_0'}^{\infty} g_2(u_1,u_2)\,\mathrm{L}_{t_1}^{\alpha}\mathrm{E}_{t_2}\left[\mathrm{H}_{\rho_1(u_1),\rho_2(u_2)}(t_1,t_2)\cdot g_1(t_1,t_2,\rho_1(u_1),\rho_2(u_2))\right]\nabla u_1\nabla u_2$$

From Theorem 7 , we get,

$$= \int_{t_0}^{\infty}\int_{t_0'}^{\infty} g_2(u_1,u_2)\left[e_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(\rho_1(u_1),\rho_2(u_2),t_0,t_0'\right)\mathrm{L}_{t_1}^{\alpha}\mathrm{E}_{t_2}[g_1(t_1,t_2)]\right]\nabla u_1\nabla u_2$$

$$= \frac{1}{q}\mathrm{L}_{t_1}^{\alpha}\mathrm{E}_{t_2}[g_1(t_1,t_2)]\;q\int_{t_0}^{\infty}\int_{t_0'}^{\infty} g_2(u_1,u_2)\,e^{\rho_1\rho_2}_{\ominus z^{\alpha}\ominus\frac{1}{q}}\left(u_1,u_2,t_0,t_0'\right)\nabla u_1\nabla u_2$$

$$= \frac{1}{q}\mathrm{L}_{t_1}^{\alpha}\mathrm{E}_{t_2}[g_1(t_1,t_2)]\cdot\mathrm{L}_{t_1}^{\alpha}\mathrm{E}_{t_2}[g_2(t_1,t_2)].$$

□

Applications

Example 1 Consider the following dynamic equation

$$\frac{\partial^2 g(t_1,t_2)}{\nabla_1^2 t_1^2} - \frac{\partial^2 g(t_1,t_2)}{\nabla_2^2 t_2^2} + \frac{\partial g(t_1,t_2)}{\nabla_2 t_2} + 9\,g(t_1,t_2) = sin_3(t_1,t_0)$$

with initial conditions,

$$g(t_1,0) = 0,\quad \frac{\partial g(t_1,0)}{\nabla_2 t_2} = sin_3(t_1,t_0)$$

$g(0,t_2)=0,\ \frac{\partial g(0,t_2)}{\nabla_1 t_1} = 3g(t_2,0).$

Solution: Applying the α- Laplace Elzaki transform on both sides,

$$z^{2\alpha}\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] - z^{\alpha}\mathrm{E}_{t_2}[g(t_0,t_2)] - \mathrm{E}_{t_2}\left[g^{\nabla_1}(t_0,t_2)\right] - \frac{1}{q^2}\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] - q\,\mathrm{L}^{\alpha}_{t_1}\left[g^{\nabla_2}(t_1,t_0)\right] - \mathrm{L}^{\alpha}_{t_1}\left[g(t_1,t_0)\right] + \frac{1}{q}\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] - q\,\mathrm{L}^{\alpha}_{t_1}\left[g(t_1,t_0)\right] + 9\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] = \mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[sin_3(t_1,t_0)].$$

Substituting the initial conditions,

$$z^{2\alpha}\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] - z^{\alpha}\mathrm{E}_{t_2}[0] - \mathrm{E}_{t_2}[3g(t_2,0)] - \frac{1}{q^2}\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] - q\,\mathrm{L}^{\alpha}_{t_1}[sin_3(t_1,t_0)] - \mathrm{L}^{\alpha}_{t_1}[0] + \frac{1}{q}\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] - q\,\mathrm{L}^{\alpha}_{t_1}[0] + 9\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] = \mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[sin_3(t_1,t_0)].$$

Taking the α-Laplace transform and Elzaki transform, we obtain

$$z^{2\alpha}\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] - 3q^3 - \frac{1}{q^2}\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] - \frac{3q}{z^{2\alpha}+9} + \frac{1}{q}\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] + 9\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] = \frac{3q^2}{z^{2\alpha}+9}.$$

On simplification, we get

$$\left(z^{2\alpha} - \frac{1}{q^2} + \frac{1}{q} + 9\right)\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] = \frac{3q^2}{z^{2\alpha}+9} - \frac{3q}{z^{2\alpha}+9} + 3q^3.$$

Which gives,

$$\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] = \frac{3q^3}{z^{2\alpha}+9}.$$

On taking inverse transform, we get

$g(t_1,t_2) = t_2 sin_3(t_1,0).$

□

Example 2 Consider the following dynamic equation

$$\frac{\partial^2 g(t_1,t_2)}{\nabla_1^2 t_1^2} + 2\frac{\partial^2 g(t_1,t_2)}{\nabla_2^2 t_2^2} + 3\frac{\partial g(t_1,t_2)}{\nabla_2 t_2} = 0$$

with initial conditions,

$g(t_1,0)=0,\ \frac{\partial g(t_1,0)}{\nabla_2 t_2} = e_{\ominus 3}(t_1,t_0)$

$g(0,t_2)=g_1(t_2,0),\ \frac{\partial g(0,t_2)}{\nabla_1 t_1} = -3g_1(t_2,0).$

Solution: Applying α- Laplace Elzaki transform on both sides,

$$z^{2\alpha}\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] - z^{\alpha}\mathrm{E}_{t_2}[g(t_0,t_2)] - \mathrm{E}_{t_2}\left[g^{\nabla_1}(t_0,t_2)\right] - \frac{2}{q^2}\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] - 2q\,\mathrm{L}^{\alpha}_{t_1}\left[g^{\nabla_2}(t_1,t_0)\right] - 2\mathrm{L}^{\alpha}_{t_1}\left[g(t_1,t_0)\right] + 3z^{\alpha}\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] - 3\mathrm{E}_{t_2}[g(t_0,t_2)] = 0.$$

On substituting given conditions,

$$z^{2\alpha}\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] - z^{\alpha}\mathrm{E}_{t_2}[g_1(t_2,0)] + 3\mathrm{E}_{t_2}[g_1(t_2,0)] + \frac{2}{q^2}\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] - 2q\,\mathrm{L}^{\alpha}_{t_1}[e_{\ominus 3}(t_1,t_0)] - 2\mathrm{L}^{\alpha}_{t_1}[0] + 3z^{\alpha}\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] - 3\mathrm{E}_{t_2}[g_1(t_2,0)] = 0.$$

Taking α-Laplace transform and Elzaki transform, we obtain

$$z^{2\alpha}\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] - z^{\alpha}q^3 + 3q^3 + \frac{2}{q^2}\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] - \frac{2q}{z^{\alpha}+3} - 2\mathrm{L}^{\alpha}_{t_1}[0] + 3z^{\alpha}\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] - 3q^3 = 0.$$

On simplification, we get

$$\left(z^{2\alpha} + \frac{2}{q^2} + 3z^{\alpha}\right)\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] = z^{\alpha}q^3 - 3q^3 + \frac{2q}{z^{\alpha}+3} + 3q^3.$$

Which gives,

$$\mathrm{L}^{\alpha}_{t_1}\mathrm{E}_{t_2}[g(t_1,t_2)] = \frac{q^3}{z^{\alpha}+3}.$$

On taking inverse transform, we get

$g(t_1,t_2) = t_2 e_{\ominus 3}(t_1,0).$

Conclusion

This article examined the α-Laplace Elzaki integral transform on time scales, including its application to elementary functions. An existence theorem and some basic properties were established. Future work will focus on solving partial dynamic equations on time scales.

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