

M* QUASI PARANORMAL COMPOSITION OPERATORS

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Abstract:

In this paper characterizes the M* quasi paranormal composition operator and M* quasi paranormal weighted composition operator on L² spaces and investigates their various properties.

Key Words:

M*quasi paranormal operator, Composition operator, Weighted composition operator.

1.Introduction

Let H be an infinite dimensional compact Hilbert space and B(H) be the algebraic of all bounded linear operators acting on H. A bounded linear operator J in a Hilbert space H is said to be M* quasi paranormal operator if it satisfies $\|J^{*2}x\|^2 \leq M \|J^3x\| \cdot \|Jx\|$, for each $x \in H$, where M is a real positive number.

1.1Preliminaries

Let $\mathcal{L}^2 = \mathcal{L}^2(\chi, \Sigma, \lambda)$ be the space of all complex valued measurable functions for which $\int_{\chi} |f|^2 d\lambda < \infty$

Let \mathcal{C}_J be a composition operator on \mathcal{L}^2 , which is caused by a non- singular measurable transformation J can be defined as $\mathcal{C}_J f = f \circ J, \forall f \in \mathcal{L}^2$. When $f \in \mathcal{L}^2$ & for any +ve value k, $\mathcal{C}_J^k f = f \circ J^k, \mathcal{C}_J^{*k} f = \int_{\chi} E(f) \circ J^{-1}$ and $\mathcal{C}_J^{*k} f = \int_{\chi} E(f) \circ J^{-k}$ where $h_k = d\lambda \circ J^{-k} / d\lambda$.

Let $W = W_{(u,J)}$ be weighted composition on \mathcal{L}^2 as $(f \rightarrow u \cdot f \circ J)$ induced by a complex valued measurable mapping u on χ & measurable transformation $J: \chi \rightarrow \chi$.

Suppose W^* is the adjoint of the weighted composition operator W and

$$W^* f = \int_{\chi} E(u \cdot f) \circ J^{-1} W^k f = u_k \cdot f \circ J^k$$

$$W^{*k}f = h_k, E(u_k, f) \circ J^{-k}$$

$$W^*Wf = h, E(u^2)J^{-1}$$

$$WW^*f = u(h, J).E(u, f), \forall f \in \mathcal{L}^2$$

2. M^* Quasi Paranormal Composition Operators

Characterization of M^* Quasi Paranormal Composition Operator.

For all f_0 which is a bounded complex valued measurable function induces the bounded operator M_{f_0} on $\mathcal{L}^2(\lambda)$, defined as $M_{f_0}f = f_0f, \forall f \in \mathcal{L}^2(\lambda)$.

Furthermore, $C^*C = M_{f_0}$ and $C^2C^{*2} = M_{f_0(2)}$ [11]

Lemma:2.1

[8] Let \mathcal{P} be the projection of L^2 on $\overline{R(C)}$, then $C^*Cf = f_0f$ and $C^*Cf = (f_0 \cdot J)Pf, \forall f \in \mathcal{L}^2(\lambda)$ where \mathcal{P} is the projection L^2 onto $\overline{R(C)}$ & $\overline{R(C)} = \{f \in \mathcal{L}^2: f \text{ is } J^{-1}\Sigma \text{ measurable}\}$.

Theorem:2.2

If $C \in B(\mathcal{L}^2(\lambda))$, then C is of M^* quasi paranormal iff $M^2f_0^{(3)} - 2\lambda f_0^{(2)} + \lambda^2 f_0 \geq 0$, a.e.

Proof

Assume $C \in B(\mathcal{L}^2(\lambda))$. When C is of M^* quasi paranormal iff

$$M^2C^{*3}C^3 - 2\lambda C^2C^{*2} + \lambda^2 C^*C \geq 0$$

$$\text{Thus, } \langle (M^2C^{*3}C^3 - 2\lambda C^2C^{*2} + \lambda^2 C^*C)X_E, X_E \rangle \geq 0$$

As each characteristic function X_E in $\Sigma \ni \lambda(E) < \infty$

Since, $C^{*3}C^3 = M_{f_0(3)}$ [11], $C^2C^{*2} = M_{f_0(2)}$, $C^*C = M_{f_0}$ [4], we have

$$\langle (M^2M_{f_0(3)} - 2\lambda M_{f_0(2)} + \lambda^2 M_{f_0})X_E, X_E \rangle \geq 0$$

$$\int_E M^2f_0^{(3)} - 2\lambda f_0^{(2)} + \lambda^2 f_0 \geq 0, \forall E \text{ in } \Sigma.$$

Thus, C is of M^* quasi paranormal iff $M^2f_0^{(3)} - 2\lambda f_0^{(2)} + \lambda^2 f_0 \geq 0$ a.e.

Corollary:2.3

Suppose that $C \in B(\mathcal{L}^2(\lambda))$ with dense range. When C is of M^* Quasi Paranormal iff

$$M^2f_0^{(3)} - 2\lambda f_0^{(2)} + \lambda^2 f_0 \geq 0$$

Example:2.4

If $X = \mathbb{N}$, where \mathbb{N} is the collection of all natural numbers & λ is the counting measure on it.

Describe $J: \mathbb{N} \rightarrow \mathbb{N}$ as $J(1) = 1, J(n + m + 1) = n, m = 0, 1, 2 \dots$ and $n \in \mathbb{N}$

$\therefore M^2 f_0^{(3)} - 2\lambda f_0^{(2)} + \lambda^2 f_0 \geq 0$, \mathcal{C} is of M^* quasi paranormal composition.

Theorem:2.5

If $\mathcal{C} \in B(\mathcal{L}^2(\lambda))$, then $\mathcal{C}^* \in M^*$ quasi paranormal iff

$$[M^2(f_0 \circ J)^{(3)}P] - 2\lambda[(f_0 \circ J)^{(2)}P] + \lambda^2[(f_0 \circ J)P] \geq 0, \text{ a.e. where } P \text{ is the projection of } \mathcal{L}^2 \text{ onto } \overline{R(\mathcal{C})}.$$

Proof:

Let \mathcal{C}^* be M^* quasi paranormal iff $M^2 \mathcal{C}^{*3} \mathcal{C}^3 - 2\lambda \mathcal{C}^2 \mathcal{C}^{*2} + \lambda^2 \mathcal{C} \mathcal{C}^* \geq 0$

i.e. $\langle (M^2 \mathcal{C}^{*3} \mathcal{C}^3 - 2\lambda \mathcal{C}^2 \mathcal{C}^{*2} + \lambda^2 \mathcal{C} \mathcal{C}^*)f, f \rangle \geq 0 \forall f \in \mathcal{L}^2$

We have $\langle \mathcal{C}^* \mathcal{C} f, f \rangle = \langle (f_0 \circ J)P f, f \rangle$, here P is the projection of \mathcal{L}^2 onto $\overline{R(\mathcal{C})}$.

Thus \mathcal{C}^* is of M^* quasi paranormal iff

$$\langle M^2[(f_0 \circ J)^{(3)}P] - 2\lambda[(f_0 \circ J)^{(2)}P] + \lambda^2[(f_0 \circ J)P]f, f \rangle \geq 0, \text{ a.e. } \forall f \in \mathcal{L}^2,$$

i.e. $M^2[(f_0 \circ J)^{(3)}P] - 2\lambda[(f_0 \circ J)^{(2)}P] + \lambda^2[(f_0 \circ J)P] \geq 0, \text{ a.e.}$

Theorem:2.6

If $\mathcal{C}^* \in B(\mathcal{L}^2(\lambda))$ with dense range, then $\mathcal{C}^* \in M^*$ quasi paranormal iff

$$M^2[(f_0 \circ J)^{(3)}P] - 2\lambda[(f_0 \circ J)^{(2)}P] + \lambda^2[(f_0 \circ J)P] \geq 0, \text{ a.e.}$$

3. M^* Quasi Paranormal Weighted Composition Operators

Weighted composition operator caused by J is a linear transforming acting on the set of complex valued Σ measurable function f , stated as $Wf = w(f \circ J)$, w is a complex valued Σ measurable function. If $w = 1$, then W is a composition operator.

Let w_k be $w(w \circ J)(w \circ J^2) \dots (w \circ J^{k-1})$. So that $W^{k,k} f = w_{k,k} (f \circ J)^{k,k}$ [10]. To analyse the weighted M^* quasi paranormal composition in effect, from [9] the conditional expectation operator E with J is denoted by $E(. / J^{-1}\Sigma) = E(.). E(f)$, stated \forall positive measurable $f \in \mathcal{L}^p(1 \leq p)$ & is distinctively determined by the following restrictions:

(i) $E(f)$ is $J^{-1}\Sigma$ measurable.

(ii) If B is a $J^{-1}\Sigma$ measurable set and $\int_B f d\lambda$ converges then $\int_B f d\lambda = \int_B E(f) d\lambda$

Here E on \mathcal{L}^p is identity (where E is a projection operator) iff $J^{-1}\Sigma = \Sigma$.

For more details [3,6,7]

Proposition:3.1

[3] For $w \geq 0$,

(i) $W^* W f = f \circ [E(w^2)] \circ J^{-1} f$

(ii) $W W^* f = w(f \circ J) E(w f)$

Characterization of weighted M^* quasi paranormal composition operators is explained by the

theorem stated below.

Theorem: 3.2

Let W be a M^* quasi paranormal operator iff

$$M^2 f_0^{(3)} [E(w_3^2)] o J^{-3} - 2\lambda f_0^{(3)} [E(w_3^2)] o J^{-2} + \lambda^2 f_0 [E(w^2)] o J^{-1} \geq 0 \text{ a.e.}$$

Proof

Let W be a M^* quasi paranormal operator iff

$$M^2 W^{*3} W^3 - 2\lambda W^2 W^{*2} + \lambda^2 W^* W \geq 0$$

Hence, $\langle (M^2 W^{*3} W^3 - 2\lambda W^2 W^{*2} + \lambda^2 W^* W) f, f \rangle \geq 0, \forall f \in L^2$

Since $W^k f = w_k (f_0 o J^k)$ and $W^{*k} f = f_0^k [E(w_k^2)] o J^{-k}$, $W^k W^{*k} = f_0^k [E(w_k^2)] o J^{-k} f$ and we have $W^* W f = f_0 [E(w^2)] o J^{-1} f$ for $w \geq 0$ [4], and hence

$$\int_E \{M^2 f_0^{(3)} [E(w_3^2)] o J^{-3} - 2\lambda f_0^{(2)} [E(w_2^2)] o J^{-2} + \lambda^2 f_0 [E(w^2)] o J^{-1}\} d\lambda \geq 0, \forall E \in \Sigma.$$

So, $M^2 f_0^{(3)} [E(w_3^2)] o J^{-3} - 2\lambda f_0^{(2)} [E(w_2^2)] o J^{-2} + \lambda^2 f_0 [E(w^2)] o J^{-1} \geq 0 \text{ a.e.}$

Corollary:3.3

Let $J^{-1} \Sigma = \Sigma$. W be a M^* quasi paranormal iff

$$M^2 f_0^{(3)} (w_3^2) o J^{-3} - 2\lambda f_0^{(2)} (w_2^2) o J^{-2} + \lambda^2 f_0 (w^2) o J^{-1} \geq 0 \text{ a.e.}$$

\tilde{J} , Aluthge transformation of J is denoted as $\tilde{J} = |J|^{\frac{1}{2}} U |J|^{\frac{1}{2}}$ introduced in [1].

Also, we can form the family of operators $J_s: 0 < s \leq 1$ for $J_s = |J|^s U |J|^{1-s}$ [2].

For the composition operator \mathcal{C} , polar decomposition is $\mathcal{C} = U |\mathcal{C}|$ where $|\mathcal{C}| f = \sqrt{f_0} f$ and

$$U f = \frac{1}{\sqrt{f_0 o J}} f o J.$$

From [2], the general Aluthge transformation for composition is

$$\mathcal{C}_s = |\mathcal{C}|^s U |\mathcal{C}|^{1-s} \text{ and } \mathcal{C}_s f = \left(\frac{f_0}{f_0 o J} \right)^{\frac{s}{2}} f o J.$$

Thus, \mathcal{C}_s is the weighted composite multiplicative operator of weight $\pi = \left(\frac{1}{\sqrt{f_0 o J}} \right)^{\frac{s}{2}}$ for $0 < s < 1$

$\therefore \mathcal{C}_s$ is weighted composition, it's simple to prove as $|\mathcal{C}_s| f = \sqrt{f_0 [E(\pi^2) o J^{-1}]} f$ &

$$|\mathcal{C}_s^*| f = v E[v f] \text{ where } v = \frac{\pi \sqrt{f_0 o J}}{[E(\pi \sqrt{f_0 o J})^2]^{\frac{1}{2}}}$$

Proposition:3.4

For every $n \in \mathbb{N}$,

$$(i) \mathcal{C}_r^k f = \pi_k (f_0 o J)^k$$

$$(ii) \mathcal{C}_r^{*k} f = f_0^{(k)} E(\pi_k f) o \mathcal{J}^{-k}$$

$$(iii) \mathcal{C}_r^{*k} \mathcal{C}_r^k f = f_0^{(k)} E(\pi_k^2) o \mathcal{J}^{-k} f$$

Corollary:3.5

Let $\mathcal{J}^{-1} \Sigma = \Sigma$, $\mathcal{C}_s \in B(\mathcal{L}^2(\lambda))$. When \mathcal{C}_s is of \mathcal{M}^* quasi paranormal iff

$$\mathcal{M}^2 f_0^{(3)}(\pi_3^2) o \mathcal{J}^{-3} - 2\lambda f_0^{(2)}(\pi_2^2) o \mathcal{J}^2 + \lambda^2 f_0(\pi^2) o \mathcal{J}^{-1} \geq 0 \text{ a. e.}$$

Proof

Since \mathcal{C}_s is \mathcal{M}^* quasi paranormal of weight $\pi = \left(\frac{f_0}{f_0 o \mathcal{J}}\right)^{\frac{s}{2}}$, then \mathcal{C}_s is of \mathcal{M}^* quasi paranormal

iff $\mathcal{M}^2 f_0^{(3)}(\pi_3^2) o \mathcal{J}^{-3} - 2\lambda f_0^{(2)}(\pi_2^2) o \mathcal{J}^{-2} + \lambda^2 f_0(\pi^2) o \mathcal{J}^{-1} \geq 0 \text{ a. e.}$ Since, B.P.Duggle[6] gave

the 2nd Aluthge transformation as $\hat{\mathcal{J}} = |\hat{\mathcal{J}}|^{\frac{1}{2}} V |\hat{\mathcal{J}}|^{\frac{1}{2}}$ and $|\hat{\mathcal{J}}| = V |\hat{\mathcal{J}}|$ is the polar decomposition of $\hat{\mathcal{J}}$.

From [12], $\hat{\mathcal{C}} = |\mathcal{C}_s|^{\frac{1}{2}} V |\mathcal{C}_s|^{\frac{1}{2}}$ and $\mathcal{C}_s = V |\mathcal{C}_s|$; $0 < s < 1$ is the polar decomposition of generalized

Aluthge transformation \mathcal{C}_s ; $0 < s < 1$ is a weighted composition of weight $w' = J^{\frac{1}{4}} \pi \left(\frac{Y_{sup} J}{J^{\frac{1}{4}}}\right) o \mathcal{J}$

where $J = f_0 E(\pi^2) o \mathcal{J}^{-1}$.

Corollary:3.6

If $\mathcal{J}^{-1} \Sigma = \Sigma$, $\hat{\mathcal{C}} \in B(\mathcal{L}^2(\lambda))$. When $\hat{\mathcal{C}}$ is of \mathcal{M}^* quasi paranormal iff

$$\mathcal{M}^2 f_0^{(3)}(w'^2_3) o \mathcal{J}^{-3} - 2\lambda f_0^{(2)}(w'^2_2) o \mathcal{J}^2 + \lambda^2 f_0(w'^2) o \mathcal{J}^{-1} \geq 0 \text{ a. e.}$$

Proof

Let $\hat{\mathcal{C}}$ is a weighted composition with weight $w' = J^{\frac{1}{4}} \pi \left(\frac{Y_{sup} J}{J^{\frac{1}{4}}}\right) o \mathcal{J}$, from this $\hat{\mathcal{C}}$ is of \mathcal{M}^* quasi paranormal iff

$$\mathcal{M}^2 f_0^{(3)}(w'^2_3) o \mathcal{J}^{-3} - 2\lambda f_0^{(2)}(w'^2_2) o \mathcal{J}^2 + \lambda^2 f_0(w'^2) o \mathcal{J}^{-1} \geq 0 \text{ a. e.}$$

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