

OSCILLATORY BEHAVIOR OF THIRD ORDER ADVANCED DIFFERENCE EQUATIONS

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Abstract

In this paper, we provide sufficient conditions for the third-order advanced difference equations of the form

$\Delta(d(\xi)\Delta(r(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))) - h(\xi)x(\xi + 1) - q(\xi)f(x(\sigma(\xi))) = 0; \xi \geq \xi_0$ to be oscillatory or to have property-B. In contrast to the current results, we established sufficient conditions for all solutions of the studied equation to be oscillatory. We provide examples to illustrate the results.

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1. Introduction

In this paper, we investigate the oscillatory behavior of third-order advanced difference equations of the form

$$\Delta(d(\xi)\Delta(r(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))) - h(\xi)x(\xi + 1) - q(\xi)f(x(\sigma(\xi))) = 0; \xi \geq \xi_0 \tag{1.1}$$

where n_0 is a non-negative integer and we assuming the following conditions hold

[H₁] $\{d(\xi)\}, \{r(\xi)\}, \{h(\xi)\}$ and $\{q(\xi)\}$ are positive real sequences for $\xi \in N$ and

$0 \leq p(\xi) \leq p_0 < \infty;$

[H₂] $\{\sigma(\xi)\}$ is an increasing sequence such that $\sigma(\xi) \geq \xi + 1$ for all $\xi \geq \xi_0;$

[H₃] $\tau(\xi) \geq \tau_0 > 0; \tau(\xi)$ is a sequence of positive integers for all $\xi \geq \xi_0;$

[H₄] $\tau \circ \sigma = \sigma \circ \tau;$

[H₅] f is continuous, non-decreasing real-valued function such that $af(a) > 0$ for $a \neq 0$ and $f(ab) \geq f(a)f(b)$ for $ab > 0;$

[H₆] There exists ξ which is positive integer satisfies $\frac{f(\xi)}{\xi} > 1$ for all $\xi > 1;$

A solution of equation (1.1), we mean a real sequence $\{x(\xi)\}$ that satisfies (1.1) for all $\xi \geq \xi_0$. A non-trivial solution of (1.1) is said to be oscillatory, if it is neither eventually positive nor eventually negative, otherwise it is non-oscillatory. A difference equation is said to be oscillatory (non-oscillatory) if all of its solutions are oscillatory (non-oscillatory).

The study of oscillatory behavior of difference equations has received considerable attention over the past decades due to its theoretical significance and applications. A number of works have addressed oscillation criteria for various classes of difference equations; see, for instance, [1]–[3],[5],[10],[14]. In particular, the oscillatory and asymptotic properties of third-order difference

equations have attracted increasing interest among researchers [4],[7]–[9],[12]–[15]. Several authors have developed oscillation results for third-order equations by employing comparison principles, averaging techniques, and related tools .[[3],[6],[11]–[16]] the authors used for oscillation of all solutions of third order difference equations was established with the help of comparison method and averaging technique.

2. Basic Lemmas

For the sake of simplicity, we define the following:

$$G(\xi) = \sum_{\alpha=\xi}^{\infty} \frac{1}{d(\alpha)}; w(\xi) = x(\xi) + p(\xi)x(\tau(\xi)); l(\xi) = d(\xi)G(\xi)G(\xi + 1); i(\xi) = \frac{r(\xi)}{G(\xi)}$$

$$T(\xi) = \sum_{\alpha=\xi_0}^{\xi-1} \frac{1}{l(\alpha)}; I(\xi) = \sum_{\alpha=\xi}^{\xi+1} \frac{1}{i(\alpha)}; U(\xi) = \sum_{\alpha=\xi}^{\xi-1} \frac{1}{i(\alpha)} \sum_{\beta=\xi_0}^{\alpha-1} \frac{1}{l(\beta)}; E(\xi) = \frac{1}{l(\xi)} \sum_{\alpha=\xi}^{\infty} [f(h(\alpha)) + q(\alpha)]; \quad \text{for}$$

$\alpha \geq \xi \geq \xi_1$ where $\xi \geq \xi_0$;

If $\{x(\xi)\}$ is a positive solution of (1.1) then $\{x(\xi)\}$ is also a positive solution of the canonical equation

$$\Delta(l(\xi)\Delta(i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))))) = h(\xi)x(\xi + 1) + q(\xi)f(x(\sigma(\xi))) \tag{2.1}$$

We introduce the following class of non-oscillatory (Without loss of generality we say positive) solutions which give the sign structure of possible non-oscillatory solutions to equations (2.1).

$$x(\xi) \in S_1 : \Delta(x(\xi) + p(\xi)x(\tau(\xi))) > 0; \Delta(i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))) < 0;$$

$$\Delta[l(\xi)\Delta(i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi))))] > 0; \text{and}$$

$$x(\xi) \in S_3 : \Delta(x(\xi) + p(\xi)x(\tau(\xi))) > 0; \Delta(i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))) > 0;$$

$$\Delta[l(\xi)\Delta(i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi))))] > 0;$$

eventually, for all $\xi \geq \xi_1 \geq \xi_0$. The preceding discussion clarifies the framework of potential non-oscillatory solutions of (1.1).

Lemma 2.2. *If $x(\xi) \in S_3$ is a positive solution of (2.1) and*

$$\sum_{\alpha=\xi}^{\infty} h(\alpha)U(\sigma(\alpha)) = \infty \tag{2.2}$$

$$\sum_{\alpha=\xi}^{\infty} q(\alpha)f(U(\sigma(\alpha))) = \infty \tag{2.3}$$

Then, $\frac{x(\xi) + p(\xi)x(\tau(\xi))}{U(\xi)}$ is eventually increasing

Proof: Assume that $\{x(\xi)\}$ is a positive solution of (2.1) such that $x(\xi) \in S_3$. Then, the corresponding function $w(\xi)$ satisfies

$$w(\sigma(\xi)) = x(\sigma(\xi)) + p(\sigma(\xi))x(\tau(\sigma(\xi))) \leq x(\sigma(\xi)) + p_0x(\sigma(\tau_0)).$$

First we claim that (2.2) and (2.3) implies

$$\lim_{\xi \rightarrow \infty} l(\xi)\Delta(i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))) = \infty \tag{2.4}$$

If not, then $\lim_{\xi \rightarrow \infty} l(\xi)\Delta(i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))) = 2M, M > 0$

and since, $l(\xi)\Delta(i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi))))$ is decreasing. We have

$$l(\xi)\Delta(i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))) > M \text{ eventually.}$$

Summing up the last inequality, we get

$$x(\xi) + p(\xi)x(\tau(\xi)) \geq MU(\xi) \text{ then,}$$

$$f(x(\sigma(\xi))) \geq f(M)f(U(\sigma(\xi))) \tag{2.5}$$

Now summing equation (2.1) from ξ_0 to ∞ , and using equation (2.5) leads to

$$M \sum_{\alpha=\xi}^{\infty} [h(\alpha)U(\sigma(\alpha))] + f(M) \sum_{\alpha=\xi}^{\infty} [q(\alpha)f(U(\sigma(\alpha)))] \leq 2M$$

This contradicts (2.2) and (2.3), hence our claim has been proven.

Next using $l(\xi)\Delta(i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi))))$ to be increasing, we see that for all $\xi \geq \xi_1 \geq \xi_0$,

$$\begin{aligned} i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi))) &= i(\xi_1)\Delta(x(\xi_1) + p(\xi_1)x(\tau(\xi_1))) + \sum_{\alpha=\xi}^{\xi-1} \frac{l(\alpha)\Delta(i(\alpha)\Delta(x(\alpha) + p(\alpha)x(\tau(\alpha))))}{l(\alpha)} \\ &\leq i(\xi_1)\Delta(x(\xi_1) + p(\xi_1)x(\tau(\xi_1))) + l(\xi)\Delta(i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))) \sum_{\alpha=\xi}^{\xi-1} \frac{1}{l(\alpha)} \end{aligned}$$

By (2.4), this implies

$$i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi))) = i(\xi_1)\Delta(x(\xi_1) + p(\xi_1)x(\tau(\xi_1))) + l(\xi)\Delta(i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))) \sum_{\alpha=\xi_0}^{\xi-1} \frac{1}{l(\alpha)} \text{ for}$$

all ξ , let us say $\xi \geq \xi_2 \geq \xi_1$. Hence,

$$\Delta\left(\frac{i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))}{T(\xi)}\right) = \frac{T(\xi)l(\xi)\Delta(i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))) - i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))}{l(\xi)T(\xi)T(\xi+1)} > 0$$

which means that $\Delta\left(\frac{i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))}{T(\xi)}\right)$ is increasing for $\xi \geq \xi_2 \geq \xi_0$.

From this we obtain

$$\begin{aligned} x(\xi) &= x(\xi_2) + p(\xi_2)x(\tau(\xi_2)) - p(\xi)x(\tau(\xi)) + \sum_{\alpha=\xi_2}^{\xi-1} \frac{T(\alpha)i(\alpha)\Delta(x(\alpha) + p(\alpha)x(\tau(\alpha)))}{T(\alpha)i(\alpha)} \\ &\leq x(\xi_2) + p(\xi_2)x(\tau(\xi_2)) - p(\xi)x(\tau(\xi)) + \frac{i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))}{T(\xi)} \sum_{\alpha=\xi_2}^{\xi-1} \frac{T(\alpha)}{i(\alpha)} \\ &\quad + \frac{i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))}{T(\xi)} \sum_{\alpha=\xi_2}^{\xi-1} \frac{T(\alpha)}{i(\alpha)} \end{aligned}$$

(2.6)

On the other hand, by Discrete L'Hospital rule

$$\lim_{\xi \rightarrow \infty} \frac{i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))}{T(\xi)} = \lim_{\xi \rightarrow \infty} l(\xi)\Delta(i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))) = \infty$$

and so by (2.6), there exists $\xi_3 > \xi_2$ such that

$$x(\xi) \leq \frac{i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))}{T(\xi)} \sum_{\alpha=\xi_0}^{\xi-1} \frac{T(\alpha)}{i(\alpha)}, \quad \xi \geq \xi_3$$

Hence,

$$\Delta\left(\frac{x(\xi) + p(\xi)x(\tau(\xi))}{U(\xi)}\right) = \frac{U(\xi)i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi))) - x(\xi) + p(\xi)x(\tau(\xi))T(\xi)}{i(\xi)U(\xi)U(\xi+1)} \geq 0$$

which implies $\frac{x(\xi) + p(\xi)x(\tau(\xi))}{U(\xi)}$ is eventually increasing. The proof of the lemma is complete.

Lemma:2.3 If $x(\xi)$ is a positive solution of (2.1) such that $x(\xi) \in S_1$. Then $\frac{x(\xi) + p(\xi)x(\tau(\xi))}{I(\xi)}$ is eventually decreasing.

Proof: Assume that $x(\xi)$ is a positive solution of (2.1) such that $x(\xi) \in S_1$ for all $\xi \geq \xi_0$.

Then $i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))$ is decreasing and

$$x(\xi) \geq \sum_{\alpha=\xi_0}^{\xi-1} \frac{i(\alpha)\Delta(x(\alpha) + p(\alpha)x(\tau(\alpha)))}{i(\alpha)} \geq i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi))) \sum_{\alpha=\xi_0}^{\xi-1} \frac{1}{i(\alpha)}$$

This implies

$$\Delta\left(\frac{x(\xi) + p(\xi)x(\tau(\xi))}{I(\xi)}\right) = \frac{I(\xi)i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi))) - x(\xi) + p(\xi)x(\tau(\xi))}{i(\xi)I(\xi)I(\xi+1)} \leq 0, \quad \xi \geq \xi_0$$

which implies $\frac{x(\xi) + p(\xi)x(\tau(\xi))}{I(\xi)}$ is eventually decreasing. This completes the proof.

Lemma 2.4 : Assume that (2.2) and (2.3) holds and

$$\lim_{\xi \rightarrow \infty} \frac{\xi}{f(\xi)} = M_3 < \infty$$

$$\text{If } \limsup_{\xi \rightarrow \infty} \left\{ \frac{1}{U(\sigma(\xi))} \sum_{\alpha=\xi}^{2\xi} \frac{1}{i(\alpha)} \sum_{\beta=\alpha}^{\alpha+1} \frac{1}{I(\beta)} \sum_{\gamma=\beta}^{\beta+1} [(f(h(\gamma)) + q(\gamma))f(U(\sigma(\gamma)))] \right\} > M_3 \tag{2.7}$$

then the class S_3 is empty for (1.1).

Proof: Let $\{x(\xi)\}$ be an eventually positive solution of (1.1). By corollary 2.1, $\{x(\xi)\}$ is also a positive solution of (2.1) and assume that $x(\xi) \in S_3$ for all $\xi \geq \xi_0$. Summing (2.1) from ξ to $k+1$, we get

$$\Delta(i(k)\Delta(x(k) + p(k)x(\tau(k)))) \geq f\left(\frac{x(\sigma(\xi))}{U(\sigma(\xi))}\right) \frac{1}{I(k)} \sum_{\alpha=\xi}^{k+1} [(f(h(\alpha)) + q(\alpha))f(U(\sigma(\alpha)))]$$

Again summing, we get

$$x(k) + p(k)x(\tau(k)) \geq f\left(\frac{x(\sigma(\xi))}{U(\sigma(\xi))}\right) \sum_{\alpha=\xi}^{k+1} \frac{1}{i(\alpha)} \sum_{\beta=\alpha}^{\alpha+1} \frac{1}{I(\beta)} \sum_{\gamma=\beta}^{\beta+1} [(f(h(\gamma)) + q(\gamma))f(U(\sigma(\gamma)))] \text{ setting}$$

$$k = 2\xi - 1 \text{ and } y = \frac{x(\sigma(\xi))}{U(\sigma(\xi))}, \text{ we obtain}$$

$\frac{y(\xi)}{y(\sigma(\xi))} \geq \frac{1}{U(\sigma(\xi))} f\left(\frac{x(\sigma(\xi))}{U(\sigma(\xi))}\right) \sum_{\alpha=\xi}^{2\xi} \frac{1}{i(\xi)} \sum_{\beta=\alpha}^{\alpha+1} \frac{1}{I(\beta)} \sum_{\gamma=\beta}^{\beta+1} [(f(h(\gamma)) + q(\gamma))f(U(\sigma(\gamma)))]$. Taking \limsup as $\psi \rightarrow \infty$ on both sides of the inequality, we obtain a contradiction and therefore the class S_3 is empty for (2.1).

3 Main Results

We present some criteria for the class S_1 of (2.1) to be empty.

Definition 3.1 [2]: Property-B

An Equation is said to have ‘‘Property-B’’, if every solution of the equation has any of the following properties:

- $\{x(\xi)\}$ is oscillatory.
- $\{x(\xi)\}$ is weakly oscillatory (i.e) $\{x(\xi)\}$ is non oscillatory, while $\{\Delta(x(\xi))\}$ is oscillatory.
- $x(\xi)$ converges monotonically to zero as $\xi \rightarrow \infty$.

By ‘‘Property-B’’ of (2.1), We mean that every solution of (2.1) satisfies the class S_3 . In the view of corollary 2.1, equation (1.1) has Property-B.

Theorem 3.2: Assume that
$$\sum_{\alpha=\xi_0}^{\infty} \frac{1}{i(\alpha)} \sum_{\beta=\xi}^{\infty} \frac{1}{I(\beta)} \sum_{\gamma=\xi}^{\infty} [f(h(\gamma)) + q(\gamma)] = \infty$$
 (3.1)

and $\lim_{\xi \rightarrow \pm\infty} \frac{\xi}{f(\xi)} = M_1 < \infty$

If $\limsup_{\xi \rightarrow \infty} \left\{ f\left(\frac{1}{I(\sigma(\xi))}\right) \sum_{\alpha=\xi_1}^{\xi-1} f(I(\sigma(\alpha)))I(\alpha+1)E(\alpha) + \sum_{\beta=\xi}^{\sigma(\xi)-1} I(\beta+1)E(\beta) + \sum_{\beta=\sigma(\xi)}^{\infty} E(\beta) \right\} > M_1$ (3.2)

then, (1.1) has Property-B.

Proof: Let $\{x(\xi)\}$ be an eventually positive solution of (1.1). $\{x(\xi)\}$ is also a positive solution of (2.1) and $x(\xi) \in S_1$ or $x(\xi) \in S_3$ for all $\xi \geq \xi_0$. Now assuming that $x(\xi) \in S_1$ for $\xi \geq \xi_0$. Summing up (2.1) twice from ξ_0 to ∞ , we get

$$i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi))) \geq \sum_{\alpha=\xi_0}^{\infty} \frac{1}{I(\alpha)} \sum_{\beta=\xi}^{\infty} [h(\beta)x(\sigma(\beta)) + q(\beta)f(x(\sigma(\beta)))] \geq \sum_{\alpha=\xi_0}^{\infty} [E(\alpha)f(x(\sigma(\alpha)))]$$

where we used the monotonicity of $f(x(\sigma(\xi)))$. Summing up the last inequality from ξ_0 to $\xi - 1$ and then

$$x(\xi) + p(\xi)x(\tau(\xi)) \geq \sum_{\alpha=\xi_0}^{\xi-1} \frac{1}{i(\alpha)} \sum_{\beta=\xi}^{\infty} E(\beta)f(x(\sigma(\beta))) \geq \sum_{\alpha=\xi_0}^{\xi-1} I(\alpha+1)E(\alpha)f(x(\sigma(\alpha))) + I(\xi) \sum_{\beta=\xi}^{\infty} E(\beta)f(x(\sigma(\beta)))$$

Thus,

$$x(\sigma(\xi)) \geq \sum_{\alpha=\xi_0}^{\xi-1} I(\alpha+1)E(\alpha)f(x(\sigma(\alpha))) + \sum_{\beta=\xi}^{\sigma(\xi)-1} I(\beta+1)E(\beta)f(x(\sigma(\beta))) + I(\xi) \sum_{\beta=\sigma(\xi)}^{\infty} E(\beta)f(x(\sigma(\beta)))$$

employing $x(\xi) + p(\xi)x(\tau(\xi))$ is increasing and $\frac{x(\xi) + p(\xi)x(\tau(\xi))}{I(\xi)}$ is decreasing, we have

$$\frac{x(\sigma(\xi))}{f(x(\sigma(\xi)))} \geq f\left(\frac{1}{I(\sigma(\xi))}\right) \sum_{\alpha=\psi_\xi}^{\xi-1} I(\alpha+1)E(\alpha) + \sum_{\beta=\xi}^{\sigma(\xi)-1} I(\beta+1)E(\beta) + I(\sigma(\xi)) \sum_{\beta=\sigma(\xi)}^{\infty} E(\beta)$$

It follows from (3.1) that $x(\xi) \rightarrow \infty$ as $\xi \rightarrow \infty$. As $\xi \rightarrow \infty$ on both sides of the above inequality, we are now taking limit superior. We obtain a contradiction and so $x(\xi) \in S_3$. Then (1.1) has Property-B. The proof of the theorem is complete.

Theorem 3.3: Assume that $\sum_{\alpha=\xi_0}^{\infty} \frac{1}{I(\alpha)} \sum_{\beta=\xi}^{\infty} [f(h(\beta)) + q(\beta)] f(I(\sigma(\beta))) = \infty$ (3.3)

and $\lim_{\xi \rightarrow 0} \frac{\xi}{f(\xi)} = M_2 < \infty$

If

$$\limsup_{\xi \rightarrow \infty} \left\{ \left(\frac{1}{I(\sigma(\xi))} \right) \sum_{\alpha=\xi_0}^{\xi-1} f(I(\sigma(\alpha))) I(\alpha+1) E(\alpha) + \frac{f(I(\sigma(\psi)))}{I(\sigma(\psi))} \sum_{\beta=\xi}^{\sigma(\xi)-1} I(\beta+1) E(\beta) + f(I(\sigma(\psi))) \sum_{\beta=\sigma(\xi)}^{\infty} E(\beta) \right\} > M_2$$

(3.4)

then, (1.1) has Property-B.

Proof: Let $\{x(\xi)\}$ be an eventually positive solution of (1.1). $\{x(\xi)\}$ is also a positive solution of (2.1) and $x(\xi) \in S_1$ or $x(\xi) \in S_3$ for all $\xi \geq \xi_0$. Now assume that $x(\xi) \in S_1$ for $\xi \geq \xi_0$. By lemma 2.3, we see that $\frac{x(\xi) + p(\xi)x(\tau(\xi))}{I(\xi)}$ is decreasing and we shall prove that (3.3) implies

$$\lim_{\xi \rightarrow \infty} \left(\frac{x(\xi) + p(\xi)x(\tau(\xi))}{I(\xi)} \right) = 0$$

If not, assume that $\lim_{\xi \rightarrow \infty} \left(\frac{x(\xi) + p(\xi)x(\tau(\xi))}{I(\xi)} \right) = L > 0$. Then, $\left(\frac{x(\xi) + p(\xi)x(\tau(\xi))}{I(\xi)} \right) \geq L$

and consequently, $f(x(\sigma(\xi))) = f(L)f(I(\sigma(\xi)))$

Summing the equation (2.1) twice from ξ_0 to ∞

$$\begin{aligned} i(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi))) &\geq \sum_{\alpha=\xi_0}^{\infty} \frac{1}{I(\alpha)} \sum_{\beta=\xi}^{\infty} [h(\beta)x(\sigma(\beta)) + q(\beta)f(x(\sigma(\beta)))] \\ &\geq f(L) \sum_{\alpha=\xi_0}^{\infty} \frac{1}{I(\alpha)} \sum_{\beta=\xi}^{\infty} [h(\beta)x(\sigma(\beta)) + q(\beta)f(x(\sigma(\beta)))] \end{aligned}$$

which contradicts (3.3) and we conclude that (3.5) holds.

Let $H(\xi) = \frac{x(\xi) + p(\xi)x(\tau(\xi))}{I(\sigma(\xi))}$. Thus

$$\frac{H(\xi)}{f(H(\xi))} \geq \left(\frac{1}{I(\sigma(\xi))} \right) \sum_{\alpha=\xi_0}^{\xi-1} f(I(\sigma(\alpha))) I(\alpha+1) E(\alpha) + \frac{f(I(\sigma(\xi)))}{I(\sigma(\xi))} \sum_{\beta=\xi}^{\sigma(\xi)-1} I(\beta+1) E(\beta) + f(I(\sigma(\xi))) \sum_{\beta=\sigma(\xi)}^{\infty} E(\beta)$$

Taking \limsup as $\xi \rightarrow \infty$ on both sides of the inequality, we obtain a contradiction and so $x(\xi) \in S_3$. Therefore (1.1) has Property-B. This completes the proof.

In the following, we apply the criteria obtained to the equation:

$$\Delta(d(\xi)\Delta(r(\xi)\Delta(x(\xi) + p(\xi)x(\tau(\xi)))) - h(\xi)x(\xi + 1) - q(\xi)x^\delta(\sigma(\xi))) \tag{3.5}$$

where δ is a ratio the odd positive integers

Corollary 3.4: Assume $\delta > 1$ and (3.1) holds. If

$$\limsup_{\xi \rightarrow \infty} \left\{ I^{-\delta}(\sigma(\xi)) \sum_{\alpha=\xi_1}^{\xi-1} I^\delta(\sigma(\alpha))I(\alpha+1)E(\alpha) + \sum_{\beta=\xi}^{\sigma(\xi)-1} I(\beta+1)E(\beta) + I(\sigma(\psi)) \sum_{\beta=\sigma(\xi)}^{\infty} E(\beta) \right\} > 0 \tag{3.6}$$

then, (3.5) has Property-B.

Corollary 3.5: Assume $\delta = 1$ and (3.1) holds. If

$$\limsup_{\xi \rightarrow \infty} \left\{ \frac{1}{I(\sigma(\xi))} \sum_{\alpha=\xi_1}^{\xi-1} I(\sigma(\alpha))I(\alpha+1)E(\alpha) + \sum_{\beta=\xi}^{\sigma(\xi)-1} I(\beta+1)E(\beta) + I(\sigma(\psi)) \sum_{\beta=\sigma(\xi)}^{\infty} E(\beta) \right\} > 1 \tag{3.7}$$

then, (3.5) has Property-B.

Corollary 3.6: Assume $\delta < 1$ and (3.1) holds. If

$$\limsup_{\xi \rightarrow \infty} \left\{ \frac{1}{I(\sigma(\xi))} \sum_{\alpha=\xi_1}^{\xi-1} I^\delta(\sigma(\alpha))I(\alpha+1)E(\alpha) + \frac{I^\delta(\sigma(\alpha))}{I(\sigma(\alpha))} \sum_{\beta=\xi}^{\sigma(\xi)-1} I(\beta+1)E(\beta) + I^\delta(\sigma(\alpha)) \sum_{\beta=\sigma(\xi)}^{\infty} E(\beta) \right\} > 1 \tag{3.8}$$

then, (3.5) has Property-B.

Theorem 3.7 :

Assume that all the assumptions of Theorem 3.2(Theorem 3.3) and Lemma 2.4 hold and Combining the criteria for both class S1 and S3 being empty. We get the oscillation of (1.1). This completes the proof.

4 Examples

Example 4.1: Consider the third-order advanced difference equation

$$\left[\Delta \left(\frac{1}{\xi} \Delta \left(\frac{1}{\xi+1} \Delta(x(\xi) + p(\xi) + x(\tau(\xi))) \right) \right) \right] - \frac{7\xi^2 + 27\xi + 14}{\xi(\xi+1)(\xi+2)(\xi+3)} x(\xi+1) - \frac{9\xi^2 + 29\xi + 22}{\xi(\xi+1)(\xi+2)(\xi+3)} f(x(\xi+1)) = 0 \tag{4.1}$$

$\xi \geq 1$

All the conditions of Theorem 3.7 are satisfied. Hence every solution of (4.1) is $x(\xi) = (-1)^\xi$. Therefore the equation (4.1) is oscillatory.

Example 4.2: Consider the third-order advanced difference equation

$$\Delta^2 [\xi \Delta(x(\xi) + p(\xi)x(\tau(\xi)))] - \left(\frac{9\xi(-1)^\xi + 11(-1)^\xi - 7\xi - 5}{1 + (-1)^\xi} \right) x(\xi+1) - (7\xi + 5)f(x(\xi+1)) = 0 \tag{4.2}$$

All the conditions of Theorem 3.7 are satisfied. Hence every solution of (4.2) is $x(\xi) = [1 - (-1)^\xi]$. Therefore the equation (4.2) is oscillatory.

Example 4.3: Consider the third-order advanced difference equation

$$\Delta \left[\xi(\xi+1) \Delta \left(\frac{1}{\xi} \Delta(x(\xi) + p(\xi)x(\tau(\xi))) \right) \right] - \frac{1}{\xi} x(\xi+1) + \left(\frac{1}{\xi} - \frac{b}{\xi(\xi+1)} \right) f(x(\sigma(\xi))), \xi > 1 \quad (4.4)$$

A simple calculation shows that $G(\xi) = \frac{1}{\xi}; l(\xi) = 1, i(\xi) = 1, \sigma(\xi) = 2\xi; f(x) = x,$

$I(\xi) = \xi, I(\sigma(\xi)) = 2\xi, U(\xi) \approx \frac{\xi^2}{2}, E(\xi) \approx \frac{b}{\xi(\xi+1)}$. Condition (3.1) holds for $b > 0$. Then (3.7)

becomes

$$\limsup_{\xi \rightarrow \infty} \left\{ \frac{1}{2^\xi} \sum_{\alpha=\xi}^{\xi-1} 2\alpha(\alpha+1) \frac{b}{\alpha(\alpha+1)} + \sum_{\beta=\alpha}^{2\xi-1} (\beta+1) \frac{b}{\beta(\beta+1)} + 2\xi \sum_{\beta=2\xi}^{\infty} \frac{b}{\beta(\beta+1)} \right\} = b+1 > 1$$

Hence, the solution of equation (4.4) converges monotonically as $\xi \rightarrow \infty$.

Example 4.4: Consider the third-order advanced difference equation

$$\Delta \left[2^{\xi+1} \Delta \left(\frac{1}{4^\xi} \Delta(x(\xi) + p(\xi)x(\tau(\xi))) \right) \right] - \frac{2b}{4^\xi} x(\xi+1) + \left(\frac{4b}{4^\xi} \right) x^2(\xi+2), \xi \geq 1 \quad (4.5)$$

A simple calculation shows that $G(\xi) = \frac{1}{2^\xi}; l(\xi) = \frac{1}{2^\xi}, i(\xi) = \frac{1}{2^\xi}, \sigma(\xi) = \xi+2;$

$I(\xi) = 2\xi, E(\xi) \approx \frac{6b}{2^\xi}$. Condition (3.1) holds for $b > 0$. Then (3.6) becomes

$$\limsup_{\xi \rightarrow \infty} \left\{ \frac{1}{2^{2\xi+4}} \sum_{\alpha=\xi}^{\xi-1} (2^{2\alpha+4})(2^{\alpha+1}) \frac{6b}{2^\alpha} + \sum_{\beta=\alpha}^{\xi+1} (2^{\beta+1}) \frac{6b}{2^\beta} + 2^{\xi+2} \sum_{\beta=\xi+2}^{\infty} \frac{6b}{2^\beta} \right\} = 36b > 1$$

Hence, the solution of equation (4.5) converges monotonically as $\xi \rightarrow \infty$.

Then the condition (3.11) becomes

$$\limsup_{\xi \rightarrow \infty} \left\{ \frac{1}{4^{\xi+2}} \sum_{\alpha=\xi}^{2\xi} 2^\alpha \sum_{\beta=\xi}^\alpha 2^\beta \sum_{\gamma=\xi}^\beta \frac{6b}{4^\gamma} 4^{\gamma+2} \right\} = \infty$$

Therefore the condition satisfies the theorem 3.3. Hence (4.5) is oscillatory.

Example: 4.5 Consider the third-order advanced difference equation

$$\Delta \left[2^{\xi+1} \Delta \left(\frac{1}{4^\xi} \Delta(x(\xi) + p(\xi)x(\tau(\xi))) \right) \right] - \frac{2b}{2^{\frac{4\xi}{3}}} x(\xi+1) + \left(\frac{4b}{2^{\frac{4\xi}{3}}} \right) x^{\frac{1}{3}}(\xi+2), \xi \geq 1 \quad (4.6)$$

where, $G(\xi) = \frac{1}{2^\xi}; l(\xi) = \frac{1}{2^\xi}, i(\xi) = \frac{1}{2^\xi}, q(\xi) = \frac{4b}{2^{\frac{4\xi}{3}}}; I(\xi) = 2\xi, E(\xi) \approx \frac{6b}{2^{\frac{\xi}{3}}}.$

Condition (3.1) and (3.3) holds for $b > 0$. Then (3.8) becomes

$$\limsup_{\xi \rightarrow \infty} \left\{ \frac{1}{2^{\xi+2}} \sum_{\alpha=\xi}^{\xi-1} (2^{\alpha+\frac{2}{3}})(2^{\alpha+1}) \frac{6b}{2^{\alpha/3}} + \frac{2^{\xi+\frac{2}{3}}}{2^{\xi+2}} \sum_{\beta=\alpha}^{\xi+1} (2^{\beta+1}) \frac{6b}{2^{\beta/3}} + 2^{(\xi+2)/3} \sum_{\beta=\xi+2}^{\infty} \frac{6b}{2^{\beta/3}} \right\} = 48b > 0$$

Hence, the solution of equation (4.6) converges monotonically as $\xi \rightarrow \infty$.

Then the condition (2.7) becomes

$$\limsup_{\xi \rightarrow \infty} \left\{ \frac{1}{4^{\xi+2}} \sum_{\alpha=\xi}^{2\xi} 2^\alpha \sum_{\beta=\xi}^{\alpha} 2^\beta \sum_{\gamma=\xi}^{\beta} \frac{6b}{2 \cdot 4^{\gamma/3}} 4^{\gamma+2} \right\} = \infty$$

Where $b > 0$. Therefore the condition satisfies the theorem 3.7. Hence (4.6) is oscillatory.

4 Conclusion

The conclusions produced in this study augment and generalize previous results for equation (1.1)'s specific instances. We determined the requirements for the third-order advanced difference equation's oscillation solution. The aforementioned instances highlight the importance and applicability of the demonstrated outcomes.

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